Solution to the 1st practice set

February 14, 2019

1. Verify the giving function is an explicit solution/a family of solution to the DE.

   (a) $\frac{dy}{dx} = 25 + y^2$, $y = 5\tan(5x)$.

   - SOLUTION:

$$\text{LHS} = y' = \frac{\partial}{\partial x}[5\tan(5x)] = 5 \cdot \frac{1}{\cos^2(5x)} \cdot 5 = \frac{25}{\cos^2(5x)},$$

$$\text{RHS} = 25 + y^2 = 25 + [5\tan(5x)]^2 = 25(1 + \frac{\sin^2(5x)}{\cos^2(5x)}) = \frac{25[\cos^2(5x) + \sin^2(5x)]}{\cos^2(5x)} = \frac{25}{\cos^2(5x)},$$

$$\text{LHS} = \frac{25}{\cos^2(5x)} = \text{RHS}.$$

   (b) $y' + 4xy = 8x^3$, $y = 2x^2 - 1 + C_1 e^{-2x^2}$.

   - SOLUTION:

$$y' = (2x^2 - 1 + C_1 e^{-2x^2})' = 4x - 0 + C_1 \cdot (-2 \cdot 2x)e^{-2x^2} = 4x - 4xC_1 e^{-2x^2},$$

$$\text{LHS} = y' + 4xy = (4x - 4xC_1 e^{-2x^2}) + 4x \cdot (2x^2 - 1 + C_1 e^{-2x^2}) = 8x^3 - 4xC_1 e^{-2x^2} + 4xC_1 e^{-2x^2} = 8x^3,$$

$$\text{RHS} = 8x^3 = \text{LHS}.$$

   (c) $y'' - 4y' + 4y = 0$, $y = C_1 e^{2x} + C_2 x e^{2x}$.

   - SOLUTION:

$$y' = 2C_1 e^{2x} + C_2(e^{2x} + x \cdot 2 \cdot e^{2x}) = (2C_1 + C_2)e^{2x} + 2C_2 x e^{2x},$$

(you can't combine the $(2C_1 + C_2)$ to one constant, because it is in the expression of y', not y.)

$$y'' = [(2C_1+C_2)e^{2x} + 2C_2 x e^{2x}]' = 2\cdot(2C_1+C_2)e^{2x} + 2C_2(e^{2x} + x\cdot 2\cdot e^{2x}) = (4C_1+4C_2)e^{2x} + 4C_2 x e^{2x},$$

$$\text{LHS} = y'' - 4y' + 4y = (4C_1 + 4C_2)e^{2x} + 4C_2 x e^{2x} - 4 \cdot [(2C_1 + C_2)e^{2x} + 2C_2 x e^{2x}] + 4 \cdot (C_1 e^{2x} + C_2 x e^{2x}),$$

$$= [(4C_1 + 4C_2) - 4 \cdot (2C_1 + C_2) + 4C_1]e^{2x} + (4C_2 - 4 \cdot 2C_2 + 4C_2)xe^{2x} = 0 = \text{RHS}$$

2. Determing the following statement is True or False, and briefly explain your reason.

   (a) The DE $y' + xy = \sin x$ has a two-parameter family of solution $y = C_1 x \sin x + C_2 \sin x$.

   - False. There are multiple reasons, you only need to mention one of them.
     i. Because the DE is a linear DE of order 1, so the family of solution will have only one parameter, not two.
     ii. It simply not a solution because the equation cannot satisfied since LHS $\neq$ RHS. (You need to show the calculation, it is a very simple calculation so that I skipped here.)

   (b) The solution to the DE $\frac{dy}{dx} = x(1-x)e^{-x}$ is increasing when$x > 1$.

   - Flase. When $x > 1$, x is positive, $(1-x)$ is negative, the exponential function e^{-x} is always positive, so $x(1-x)e^{-x}$ is negative, the DE implies the derivative $y' = \frac{dy}{dx} = x(1-x)e^{-x}$ is negative, so decreaing.

   (c) The function $y = \sin x$ is an explicit solution to the DE $y'' + y = 0$, and the interval $(0, 2\pi)$ is an interval of solution.

   - True. You need to mention both of the reason to show it is true.

i. Verify the solution. $y'' + y = (\sin x)'' + \sin x = (\cos x)' + \sin x = -\sin x + \sin x = 0$.

ii. The solution $y = \sin x$ is well-defined on the interval $(0, 2\pi)$.

3. Draw the phase portrait of the DE $\frac{dy}{dx} = y^2(4-y)$, and if a solution to the DE satisfies the initial condition $y(4) = 2$, then what is the range of this solution?

- SOLUTION: Since the initial value $y = 2$ falls between the equilibrium points $y = 0$ and $y = 4$, so the range of this IVP will be $(0, 4)$, aka $0 < y < 4$.

4. Find all solutions to the following differential equations by separation of variables.

(a) $\frac{dT}{dt} = kT$, where $k \neq 0$ is an constant and $T > 0$.

- SOLUTION:

$$\begin{aligned}\frac{dT}{T} &= kdt,\\ \int \frac{dT}{T} &= \int kdt,\\ \ln|T| &= kt + C,\\ \ln T &= kt + C, \text{since } T > 0,\\ T(t) &= Ce^{kt}, \quad C > 0.\end{aligned}$$

ANSWER: $\ln T(t) = kt + C$(implicit solution) OR $T(t) = Ce^{kt}, C > 0$ (explicit solution).

(b) $y' = x\sqrt{x^2+1}$.

- SOLUTION:

$$\begin{aligned}\frac{dy}{dx} &= x\sqrt{x^2+1},\\ dy &= x\sqrt{x^2+1}dx,\\ \int dy &= \int x\sqrt{x^2+1}dx,\\ y &= \frac{1}{2}(x^2+1)^{\frac{1}{2}+1} + C, (\text{u-sub}, u = x^2+1),\\ y(x) &= \frac{1}{2}(x^2+1)^{\frac{3}{2}} + C.\end{aligned}$$

ANSWER: $y(x) = \frac{1}{2}(x^2+1)^{\frac{3}{2}} + C$.

(c) $dx + e^{x+2y}dy = 0$.

- SOLUTION:

$$\begin{aligned}e^{x+2y}dy &= -dx,\\ e^{2y}dy &= e^{-x}dx,\\ \int e^{2y}dy &= \int e^{-x}dx,\\ \frac{1}{2}e^{2y} &= -e^{-x} + C,\\ e^{2y} &= -2e^{-x} + C,\\ 2y &= \ln(-2e^{-x} + C),\\ y &= \ln(-2e^{-x} + C).\end{aligned}$$

ANSWER: $\frac{1}{2}e^{2y} = -e^{-x} + C$ OR $e^{2y} = -2e^{-x} + C$ OR $e^{2y} + 2e^{-x} = C$ OR $y = \ln(-2e^{-x} + C)$.

5. Find all solutions to the following linear differential equations by multiplying an integrating factor.

(a) $\frac{dy}{dx} + 4x^3y = x^3$.

- SOLUTION: $P(x) = 4x^3$, $f(x) = x^3$, so the integrating factor is

$$\mu(x) = e^{\int P(x)dx} = e^{\int 4x^3 dx} = e^{x^4},$$

by the formula $y(x) = \mu^{-1}(x) \cdot [\int \mu(x) f(x) dx + C]$, the solution is

$$\begin{aligned}
y(x) &= \frac{\int e^{x^4} x^3 dx + C}{e^{x^4}}, \\
y(x) &= \frac{\frac{1}{4}\int 4x^3 e^{x^4} dx + C}{e^{x^4}}, (u = x^4) \\
y(x) &= \frac{1}{4} e^{-x^4}(e^{x^4} + C), \\
y(x) &= Ce^{-x^4} + \frac{1}{4}, \text{(here the constant } \frac{1}{4}C \to C)
\end{aligned}$$

ANSWER: $y(x) = Ce^{-x^4} + \frac{1}{4}$. (You can use the fomular directly!)

(b) $xy' + y = \sin 2x$.

- SOLUTION: Convert the DE to the standard form $y' + P(x)y = f(x)$.

$$y' + \frac{1}{x}y = \frac{\sin(2x)}{x} \quad \Rightarrow \quad P(x) = \frac{1}{x},\ f(x) = \frac{\sin(2x)}{x},$$

the integrating factor is

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x| = x,$$

By the formula $y(x) = \mu^{-1}(x) \cdot [\int \mu(x) f(x) dx + C]$, the solution is

$$\begin{aligned}
y(x) &= x^{-1}[\int x \cdot \frac{\sin(2x)}{x} dx + C] \\
y(x) &= x^{-1}[\int \sin(2x) dx + C] \\
y(x) &= x^{-1}[\int 2\sin x \cos x dx + C] \\
y(x) &= x^{-1}[\sin^2 x + C]
\end{aligned}$$

ANSWER: $y(x) = x^{-1}\sin^2 x + Cx^{-1}$.

6. Find all solutions to the following DEs in two ways: separation of variable, and multiplying an integrating factor.

(a) $\frac{dy}{dx} = 5y$.

- SEPERATION OF VARIABLE:

$$\begin{aligned}
\frac{dy}{y} &= 5dx, \\
\int \frac{dy}{y} &= \int 5dx, \\
\ln|y| &= 5x + C, \\
|y| &= Ce^{5x}, C > 0, \\
y &= Ce^{5x}, \text{for any } C.
\end{aligned}$$

- INTEGRATING FACTOR:

$$y' + (-5)y = 0, \quad \Rightarrow \quad P(x) = -5, f(x) = 0,$$

the integrating factor

$$\mu(x) = e^{\int P(x)dx} = e^{\int -5dx} = e^{-5x},$$

solution is

$$y(x) = \frac{\int \mu(x) f(x) dx + C}{\mu(x)} = \frac{\int e^{-5x} \cdot 0 dx + C}{e^{-5x}} = \frac{0 + C}{e^{-5x}} = Ce^{5x}.$$

(b) $y' + 3x^2y = x^2$.

- SEPERATION OF VARIABLE:

$$\begin{aligned}\frac{dy}{dx} &= x^2 - 3x^2y,\\ \frac{dy}{dx} &= x^2(1-3y),\\ \frac{dy}{1-3y} &= x^2dx,\\ -\frac{1}{3}\ln|1-3y| &= \frac{1}{3}x^3 + C,\\ \ln|1-3y| &= -x^3 + C.\end{aligned}$$

- INTEGRATING FACTOR:

$$P(x) = 3x^2, f(x) = x^2,$$

so the integrating factor

$$\mu(x) = e^{\int 3x^2dx} = e^{x^3},$$

the solution will be

$$y(x) = \frac{\int e^{x^3}\cdot x^2dx + C}{e^{x^3}} = e^{-x^3}(\frac{1}{3}e^{x^3} + C) = \frac{1}{3} + Ce^{-x^3}.$$

- REMARK: The two solutions are the same, just in different format. See calculation below,

$$\begin{aligned}\ln|1-3y| &= -x^3 + C,\\ |1-3y| &= e^Ce^{-x^3} = Ce^{-x^3}, (C > 0)\\ 1-3y &= Ce^{-x^3} \text{ for any } C,\\ y &= \frac{1}{3} - \frac{1}{3}Ce^{-x^3},\\ y &= \frac{1}{3} + Ce^{x^3}.\end{aligned}$$

7. Solve the following Initial Value Problem and point out the interval of solution:

(a) $\frac{dy}{dx} = \frac{1}{y}$, $y(0) = 1$.

- SOLUTION:

$$\begin{aligned}ydy &= dx,\\ \frac{1}{2}y^2 &= x + C,\end{aligned}$$

use initial condition to find the value of C:

$$\frac{1}{2}\cdot 1^2 = 0 + C, \quad\Rightarrow\quad C = \frac{1}{2}.$$

So the solution to the IVP is $y = \sqrt{2x+1}$, the interval of solution is $(-\frac{1}{2}, \infty)$. ($y = -\sqrt{2x+1}$ is not a solution to the IVP, because simply $y(0) = -\sqrt{0+1} = -1 \neq 1$.)

(b) $y' = -xe^y$, $y(10) = 1$.

- SOLUTION:

$$\begin{aligned}\frac{dy}{dx} &= -xe^y,\\ e^{-y}dy &= -xdx,\\ -e^{-y} &= -\frac{1}{2}x^2 + C,\end{aligned}$$

use initial condition to find the value of C:

$$-e^{-1} = -\frac{1}{2}\cdot 10^2 + C, \quad\Rightarrow\quad C = 50 - e^{-1}.$$

So the solution is

$$e^{-y} = \frac{1}{2}x^2 + e^{-1} - 50, \quad \Rightarrow \quad -y = \ln(\frac{1}{2}x^2 + e^{-1} - 50),$$

the interval of solution is $(\sqrt{100 - 2e^{-1}}, \infty)$. (We need $\frac{1}{2}x^2 + e^{-1} - 50 > 0$, so x takes value in either $(-\infty, -\sqrt{100-2e^{-1}})$ or $(\sqrt{100-2e^{-1}}, \infty)$. The initial value condition is given when $x = 10 \in (\sqrt{100-2e^{-1}}, \infty)$, so we only take the later interval and discard the first interval.)

8. Cooling of a cake: When a cake is removed from an oven, its temperature is measured at 300F. The room temerature is 65F. We know the cooling constant is $k = \frac{1}{3}\ln\frac{13}{23}$. How long will it take for the cake to cool off to a temperature of 150F? (Newton's Law of Cooling)

   - SOLUTION: Assume the temperature of the cake at time t is $T(t)$. So we can set up an initial value problem by the situation described in the question

   $$\begin{cases} \frac{dT(t)}{dt} = \frac{1}{3}\ln\frac{13}{23}(T(t) - 65), \\ T(0) = 300, \end{cases}$$

   the DE is separable,

   $$\frac{dT}{T-65} = \frac{1}{3}\ln\frac{13}{23}dt, \quad \Rightarrow \quad \ln|T-65| = \frac{1}{3}\ln\frac{13}{23}t + C, \quad \Rightarrow \quad T - 65 = Ce^{\frac{1}{3}\ln\frac{13}{23}t},$$

   use the initial condition to determine the value of the parameter C,

   $$T(0) - 65 = Ce^{\frac{1}{3}\ln\frac{13}{23}\cdot 0}, \quad \text{i.e.} \quad 300 - 65 = C \cdot 1,$$

   so $C = 235$, the solution to this IVP is

   $$T(t) = 235e^{\frac{1}{3}\ln\frac{13}{23}t},$$

   let $T(t) = 150$, we will have an equation about t,

   $$150 = 235e^{\frac{1}{3}\ln\frac{13}{23}t}, \quad \Rightarrow \quad e^{\frac{1}{3}\ln\frac{13}{23}t} = \frac{150}{235}, \quad \Rightarrow \quad \frac{1}{3}\ln\frac{13}{23}\cdot t = \ln\frac{150}{235}$$

   so

   $$t = \frac{3\ln\frac{150}{235}}{\ln\frac{13}{23}} = 2.3606\,(\text{by calculator}).$$

9. Growth and Decay: The population of a community is known to increase at a rate proportional to the number of people present at time t. If an initial population P_0 has doubled in 5 years, how long will it take to triple?

   - SOLUTION: Assume the population at time t is $P(t)$.

   $$\begin{array}{ccc} \text{increase at a rate proportional to the population at time t} & \Rightarrow & \frac{dP}{dt} = kP \\ \text{initial population is } P_0 & \Rightarrow & P(0) = P_0 \\ \text{doubled in 5 years} & \Rightarrow & P(5) = 2P_0 \end{array}$$

   solve the DE,

   $$\frac{dP}{P} = kdt, \quad \Rightarrow \quad \ln|P| = kt, \quad \Rightarrow \quad P(t) = Ce^{kt}.$$

   The conditions

   $$\begin{array}{rcccc} P(0) = P_0 & \Rightarrow & Ce^{k\cdot 0} = P_0 & \Rightarrow & C = P_0 \\ P(5) = 2P_0 & \Rightarrow & P_0e^{5k} = 2P_0 & \Rightarrow & e^{5k} = 2 \\ P(t) = 3P_0 & \Rightarrow & P_0e^{kt} = 3P_0 & \Rightarrow & \text{solve for } t: \ e^{kt} = 3 \end{array}$$

   we have

   $$\begin{array}{ccc} e^{5k} = 2 & \Rightarrow & 5k = \ln 2 \\ e^{kt} = 3 & \Rightarrow & kt = \ln 3 \end{array}$$

   so

   $$\frac{kt}{5k} = \frac{\ln 3}{\ln 2}, \quad \Rightarrow \quad t = 5\frac{\ln 3}{\ln 2} = 7.925$$

10. Challenge Questions: verify solution. (Won't count for completeness and correctness purpose!)

(a) $\frac{dy}{dx} = (y-1)(1-2y)$, $\ln\left(\frac{2y-1}{y-1}\right) = x$.

- Do differentiation on the solution equation

$$\begin{aligned}
d\left[\ln\left(\frac{2y-1}{y-1}\right)\right] &= dx \\
d[\ln(2y-1) - \ln(y-1)] &= dx \\
\frac{2}{2y-1}dy - \frac{1}{y-1}dy &= dx \\
\frac{2(y-1)-(2y-1)}{(2y-1)(y-1)}dy &= dx \\
-\frac{dy}{(2y-1)(y-1)} &= dx \\
\frac{dy}{dx} &= -(2y-1)(y-1)
\end{aligned}$$

(b) $x\frac{dy}{dx} + y = \frac{1}{y^2}$, $x^3y^3 = x^3 + 1$.

- Do differentiation on the solution equation

$$\begin{aligned}
d(x^3y^3) &= d(x^3+1) \\
d(x^3)y^3 + x^3 d(y^3) &= 3x^2 dx \\
3x^2y^3 dx + 3x^3y^2 dy &= 3x^2 dx \\
3x^2y^3 + 3x^3y^2\frac{dy}{dx} &= 3x^2 \\
\frac{1}{3x^2y^2}\left(3x^2y^3 + 3x^3y^2\frac{dy}{dx}\right) &= \frac{3x^2}{3x^2y^2} \\
x\frac{dy}{dx} + y &= \frac{1}{y^2}
\end{aligned}$$

(c) $\frac{dy}{dx} - 2xy = e^x$, $y = e^{x^2}\int_0^x e^{t-t^2}dt$.

- Do derivative by Fundamental Theorem of Calculus(If you don't reminder what it is, Google knows everything!)

$$\begin{aligned}
dy &= d(e^{x^2})\int_0^x e^{t-t^2}dt + e^{x^2}d(\int_0^x e^{t-t^2}dt), \quad \text{(product rule)} \\
dy &= 2xe^{x^2}\cdot dx \cdot \int_0^x e^{t-t^2}dt + e^{x^2}\cdot e^{x-x^2}dx \\
dy &= 2xdx\cdot\left(e^{x^2}\int_0^x e^{t-t^2}dt\right) + e^{x^2+x-x^2}dx \\
dy &= 2xdx\cdot y + e^x dx, \quad \Rightarrow \quad \frac{dy}{dx} = 2xy + e^x.
\end{aligned}$$