# Practice Sheet - Week 1 - Solution

September 8, 2019

1. Determine which of the followings are differential equations, and state your reason.

   (a) $y' = 0$ (b) $y^{(6)} - y^6 = 0$ (c) $f^2 f^{(2)} = x$ (d) $\frac{dy}{dx} = 1$ (e) $y' + x$

   • SOLUTION: (a) Yes, it is. (b) Yes, it is. (c) Yes, it is. (d)Yes, it is. (e) No, it is not even an equation.

2. Determine the order of the following differential equations. Also determine if it is linear or nonlinear.

   (a) $ty''' - y'' - 2y = 0$ (b) $\frac{dy^6}{dt^6} - 2\frac{dy}{dt} + y = t^2$ (c) $y^{(3)} + (y')^3 = x + 1$ (d) $\cos y + y' = t$

   • SOLUTION: (a) 3rd-order, linear. (b) 6th-order, linear. (c) 3rd-order (because of the term $y^{(3)}$-3rd order derivative), not linear (because of the term $(y')^3 = y' \cdot y' \cdot y'$-not linear). (d) 1st-order, not linear (because of the nonlinear term $\cos y$).

3. Find the number of parameters that the family of solution has for the following D.E.

   (a) $y'' + 2y' + 4y = 5$,
   (b) $x^3 y''' + 2x^2 y'' + 20xy' - 78y = 0$,

   • SOLUTION: (a) Its general solution will be a 2-parameter family of functions becuase it is a 2nd-order D.E. (b) Its general solution will be a 3- paramter family of functions because it is a 3rd-order D.E.

4. Consider the given 3 - parameter family function $y = C_1 e^x + C_2 e^{-x} + C_3$ , which one of the following D.E.s has the given family function as its general solution?

   (a) $y'' - y' = 0$ (b) $y'' - 1 = 0$ (c) $y''' - y' = 0$ (d) $y^3 - y = 0$

   • SOLUTION: Choice (c) is correct. Because: choices (a) and (b) are 2nd order DEs, which can't have 3-parameters of solutions; choice (d) is not a D.E.

5. Determine the following statement is True or False, and briefly explain your reason.

   The DE $y' + xy = \sin x$ has a two-parameter family of solution $y = C_1 x \sin x + C_2 \sin x$.

   • SOLUTION: False, a 1st order D.E. can't have a 2-parameter family of solutions.

6. Verify $y = 5\tan(5x)$ is an explicit solution to the D.E. $\frac{dy}{dx} = 25 + y^2$.

   • SOLUTION:

$$\text{LHS} = y' = \frac{\partial}{\partial x}[5\tan(5x)] = 5 \cdot \frac{1}{\cos^2(5x)} \cdot 5 = \frac{25}{\cos^2(5x)},$$

$$\text{RHS} = 25 + y^2 = 25 + [5\tan(5x)]^2 = 25(1 + \frac{\sin^2(5x)}{\cos^2(5x)}) = \frac{25[\cos^2(5x) + \sin^2(5x)]}{\cos^2(5x)} = \frac{25}{\cos^2(5x)},$$

$$\text{LHS} = \frac{25}{\cos^2(5x)} = \text{RHS}.$$

7. Check if the function $y(t) = t + 1$ is a solution to the following differential equation:

$$\frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t}.$$

- SOLUTION: The left hand side of the equation is

$$\begin{aligned} \text{LHS} &= \frac{d}{dt}(t+1) \\ &= 1 \end{aligned}$$

while the right hand side of the equation is

$$\begin{aligned} \text{RHS} &= \frac{y(t)^2 - 1}{t^2 + 2t} \\ &= \frac{(t+1)^2 - 1}{t^2 + 2t} = \frac{t^2 + 2t + 1 - 1}{t^2 + 2t} \\ &= \frac{t^2 + 2t}{t^2 + 2t} = 1. \end{aligned}$$

Since the $LHS = RHS$, then $y(t) = 1 + t$ is a solution to this differential equation.

8. Verify the given function is a family of solution to the D.E.

   (a) $y' + 4xy = 8x^3$, $y = 2x^2 - 1 + C_1 e^{-2x^2}$.

   (b) $y'' - 4y' + 4y = 0$, $y = C_1 e^{2x} + C_2 x e^{2x}$.

- SOLUTION(a):

$$\begin{aligned} y' &= (2x^2 - 1 + C_1 e^{-2x^2})' = 4x - 0 + C_1 \cdot (-2 \cdot 2x) e^{-2x^2} = 4x - 4xC_1 e^{-2x^2}, \\ \text{LHS} = y' + 4xy &= (4x - 4xC_1 e^{-2x^2}) + 4x \cdot (2x^2 - 1 + C_1 e^{-2x^2}) = 8x^3 - 4xC_1 e^{-2x^2} + 4xC_1 e^{-2x^2} = 8x^3, \\ \text{RHS} &= 8x^3 = \text{LHS}. \end{aligned}$$

- SOLUTION(b):

$$y' = 2C_1 e^{2x} + C_2(e^{2x} + x \cdot 2 \cdot e^{2x}) = (2C_1 + C_2)e^{2x} + 2C_2 x e^{2x},$$

(you can't combine the $(2C_1 + C_2)$ to one constant, because it is in the expression of $y'$, not $y$.)

$y'' = [(2C_1 + C_2)e^{2x} + 2C_2 x e^{2x}]' = 2 \cdot (2C_1 + C_2)e^{2x} + 2C_2(e^{2x} + x \cdot 2 \cdot e^{2x}) = (4C_1 + 4C_2)e^{2x} + 4C_2 x e^{2x}$,

$$\begin{aligned} \text{LHS} = y'' - 4y' + 4y &= (4C_1 + 4C_2)e^{2x} + 4C_2 x e^{2x} - 4 \cdot [(2C_1 + C_2)e^{2x} + 2C_2 x e^{2x}] + 4 \cdot (C_1 e^{2x} + C_2 x e^{2x}), \\ &= [(4C_1 + 4C_2) - 4 \cdot (2C_1 + C_2) + 4C_1]e^{2x} + (4C_2 - 4 \cdot 2C_2 + 4C_2)x e^{2x} = 0 = \text{RHS} \end{aligned}$$

9. Consider the initial value problem

$$\begin{cases} \dfrac{dy}{dx} = y(1-y), \\ y(0) = \dfrac{2}{3}. \end{cases}$$

We know the function $y = \frac{Ce^x}{1 + Ce^x}$ is a family of solution to the DE $y' = y(1-y)$, find an explicit solution to the I. V. P.

• SOLUTION: To find the explicit solution we only need to find the value for $C$ sucht that $y(0) = \frac{2}{3}$. The initial value is given when $x = 0$, so

$$\frac{2}{3} = y(0) = \frac{Ce^0}{1 + Ce^0} = \frac{C}{1 + C},$$
$$2(1 + C) = 3C, \quad \text{so } C = 2.$$

(note: $e^0 = 1$).

10. Rewrite the following D.E.s in differential form, and find the functions $M(x, y)$ and $N(x, y)$ in the definition.

(a) $y \cos x + (y + \sin x)y' = 0$ (b) $\frac{dy}{dx} = \frac{y}{x+y}$

• SOLUTION: (a)

$$y \cos x + (y + \sin x)\frac{dy}{dx} = 0$$
$$y \cos x dx + (y + \sin x)dy = 0$$
$$M(x, y) = y \cos x, \qquad N(x, y) = y + \sin x.$$

(b)

$$\frac{dy}{dx} = \frac{y}{x + y}$$
$$dy = \frac{y}{x + y}dx$$
$$\frac{y}{x + y}dx - dy = 0$$
$$M(x, y) = \frac{y}{x + y}, \qquad N(x, y) = -1.$$

(note: you may get other expression of differential form, for example $ydx - (x + y)dy = 0$, it is also correct).

11. Verify that the relation $x^2 + y^2 = 4$ is an implicit solution of the D.E.

$$\frac{dy}{dx} = -\frac{x}{y}.$$

• SOLUTION: The differential equation can be written in differential form

$$ydy = -xdx \qquad \Rightarrow \qquad xdx + ydy = 0 \, (y \neq 0),$$

implicit differentiate the implicite solution

$$d(x^2 + y^2) = d(4) \qquad \Rightarrow \qquad 2xdx + 2ydy = 0 \qquad \Rightarrow \qquad xdx + ydy = 0,$$

which is the differential form of the D.E., so it is a solution.

12. A box of peanut butter ice cream is removed from a freezer where the temperature is $15^o$F, and then it is placed on a table in a room where the temperature is $68^o$F. After 30 min, the temperature of the ice cream is increased to $45^o$F. By Newton's law of warming/cooling, the rate at which the temperature of a subject changes is proportional to the difference between the temperature of the subject and the tempertaure of the surrounding. Set up a math model by using differential equation.

DID NOT COVER YET, COMING NEXT WEEK.

13. The population of bacteria in a culture grows at a rate **proportional** to the number of bacteria present at time $t$. Set up a differential equation to represent the relation between the rate of grow and the population of the bacteria.

    DID NOT COVER YET, COMING NEXT WEEK.

14. Challenge Questions (Won't be on Quiz): verify solution.

    (a) $\frac{dy}{dx} = (y-1)(1-2y)$, $\ln\left(\frac{2y-1}{y-1}\right) = x$.

    (b) $x\frac{dy}{dx} + y = \frac{1}{y^2}$, $x^3y^3 = x^3 + 1$.

    (c) $\frac{dy}{dx} - 2xy = e^x$, $y = e^{x^2}\int_0^x e^{t-t^2}\,dt$.

    • SOLUTION:

    (a) $\frac{dy}{dx} = (y-1)(1-2y)$, $\ln\left(\frac{2y-1}{y-1}\right) = x$.

    Do implicit differentiation on the solution equation

$$
\begin{aligned}
d\left[\ln\left(\frac{2y-1}{y-1}\right)\right] &= dx \\
d[\ln(2y-1) - \ln(y-1)] &= dx \\
\frac{2}{2y-1}dy - \frac{1}{y-1}dy &= dx \\
\frac{2(y-1)-(2y-1)}{(2y-1)(y-1)}dy &= dx \\
-\frac{dy}{(2y-1)(y-1)} &= dx \\
\frac{dy}{dx} &= -(2y-1)(y-1)
\end{aligned}
$$

    (b) $x\frac{dy}{dx} + y = \frac{1}{y^2}$, $x^3y^3 = x^3 + 1$.

    Do implicit differentiation on the solution equation

$$
\begin{aligned}
d(x^3y^3) &= d(x^3+1) \\
d(x^3)y^3 + x^3 d(y^3) &= 3x^2dx \\
3x^2y^3dx + 3x^3y^2dy &= 3x^2dx \\
3x^2y^3 + 3x^3y^2\frac{dy}{dx} &= 3x^2 \\
\frac{1}{3x^2y^2}\left(3x^2y^3 + 3x^3y^2\frac{dy}{dx}\right) &= \frac{3x^2}{3x^2y^2} \\
x\frac{dy}{dx} + y &= \frac{1}{y^2}
\end{aligned}
$$

    (c) $\frac{dy}{dx} - 2xy = e^x$, $y = e^{x^2}\int_0^x e^{t-t^2}\,dt$.

    Do implicit derivative by Fundamental Theorem of Calculus(If you don't reminder what it is, Google knows everything!)

$$dy = d(e^{x^2})\int_0^x e^{t-t^2}dt + e^{x^2}d(\int_0^x e^{t-t^2}dt), \quad \text{(product rule)}$$

$$dy = 2xe^{x^2}\cdot dx\cdot\int_0^x e^{t-t^2}dt + e^{x^2}\cdot e^{x-x^2}dx$$

$$dy = 2xdx \cdot \left( e^{x^2} \int_0^x e^{t-t^2} dt \right) + e^{x^2+x-x^2} dx$$

$$dy = 2xdx \cdot y + e^x dx, \quad \Rightarrow \quad \frac{dy}{dx} = 2xy + e^x.$$