Practice Sheet - Week 2

September 5, 2019

1. (Question 12 from Week 1) A box of peanut butter ice cream is removed from a freezer where the temperature is 15^oF, and then it is placed on a table in a room where the temperature is 68^oF. After 30 min, the temperature of the ice cream is increased to 45^oF. By Newton's law of warming/cooling, the rate at which the temperature of a subject changes is proportional to the difference between the temperature of the subject and the tempertaure of the surrounding. Set up a math model by using differential equation.

   SOLUTION: Let us use the function $T(t)$ to denote the temperature of the ice cream at time t(declear the unknown function when the question didn't assign you one). The peanut butter ice cream is the object that the temperature we are interested in, so the temperature when it is just removed from the freezer gives the initial condition $T(0) = 15$. The room that the ice cream in is the surrounding enviroment, so $T_s = 68$. After 30 min the temperature being 45 gives another condition $T(30) = 45$. Since it is the case of warming, $T'(t) > 0$, and $(T(t) - T_s) < 0$, for the proportional constant $k > 0$, we have a math model

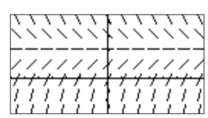
$$\begin{cases} T'(t) = -k(T(t) - 68), \\ T(0) = 15, \\ T(30) = 45. \end{cases}$$

2. (Question 13 from Week 2) The population of bacteria in a culture grows at a rate **proportional** to the number of bacteria present at time t. Set up a differential equation to represent the relation between the rate of grow and the population of the bacteria.

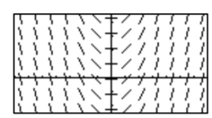
   SOLUTION: Let us use the function $P(t)$ to denote the population of the bacteria in the culture at time t, and also let k be the proportional constant for this case. Then we have D.E.

$$P'(t) = kP(t).$$

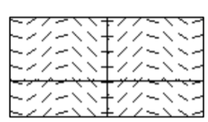
(A)



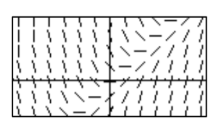
(B)



(C)



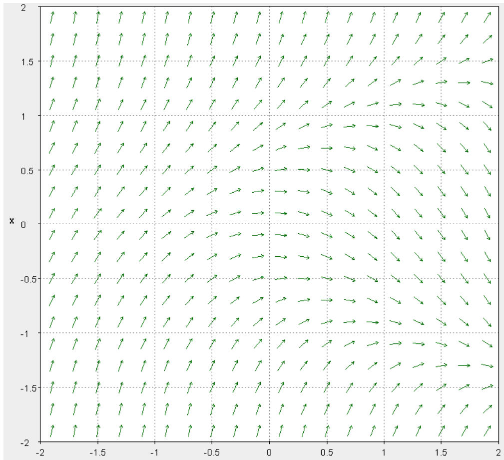
(D)



3. Match the following slope fields with their equations
   (a) $\frac{dy}{dt} = \sin t$ ———————————————— (C)
   (b) $\frac{dy}{dt} = t - y$ ———————————————— (D)
   (c) $\frac{dy}{dt} = 2 - y$ ———————————————— (A)
   (d) $\frac{dy}{dt} = t$ ———————————————— (B)

4. Suppose the following ODE

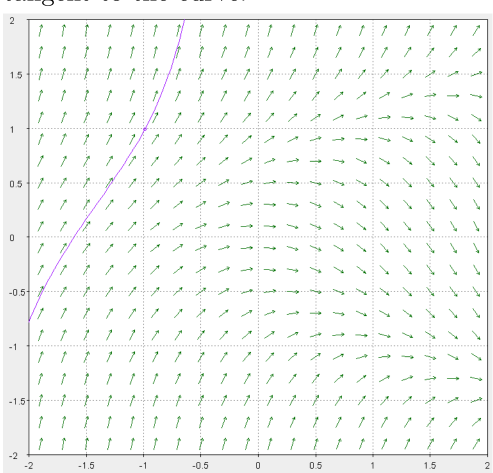
$$\frac{dy}{dt} = y^2 - t$$

has the following Slope Field: 

(a) Suppose $y(t)$ is a solution to this ODE and also you know that $y(-1) = 1$. Then based on the slope field, what is your prediction for the long term behavior of $y(t)$, that is, what is your prediction of

$$\lim_{t\to\infty} y(t) = ?$$

SOLUTION: If the solution goes through the point $y(-1) = 1$, which is the point $(-1, 1)$ in the graph, then my prediction for the solution would be the curve that flows along the direction field. To sketch the curve, point out the point $(-1, 1)$ first and then draw a curve such that the vectors on the graph are tangent to the curve.



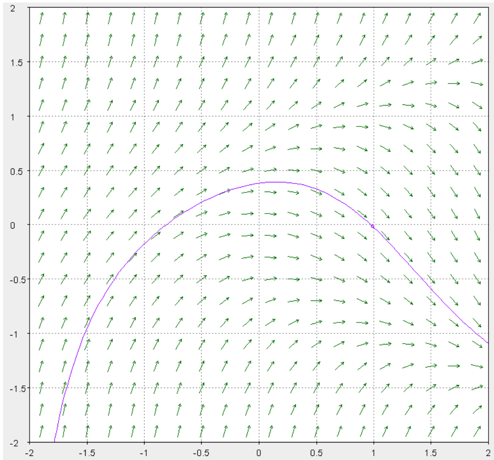
- Hence by my sketch

$$\lim_{t\to\infty} y(t) = \infty.$$

(b) Suppose $y(t)$ is a solution to this ODE and also you know that $y(1) = 0$. Then based on the slope field, what is your prediction for the long term behavior of $y(t)$, that is, what is your prediction of

$$\lim_{t\to\infty} y(t) = ?$$

SOLUTION: If the solution goes through the point $y(1) = 0$, i.e. $(1, 0)$. Repeat the drawing steps in (a), then my prediction for the solution would look like the following curve.



Using the given information, we have could say that

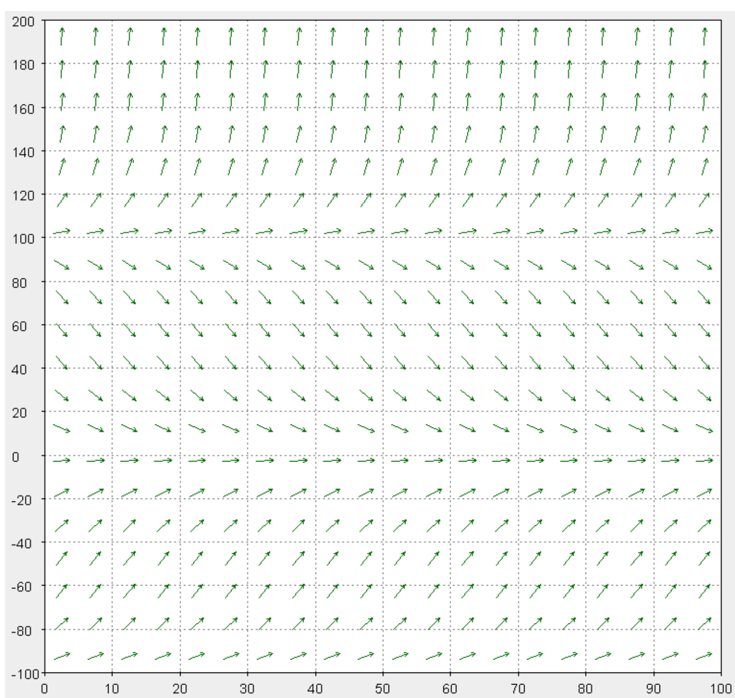
$$\lim_{t\to\infty} y(t) < -1.$$

It might be too bold to say that the limit is $-\infty$. This is because what if the solution keeps going down but then eventually goes back up? Who knows? But given the current slope field, the only prediction we can make is that it is less than -1, because the tangents above the point $(2, -1)$ all point down.

If you'd like to be more precise, it actually seems that the limit might be between -1.5 and -1.

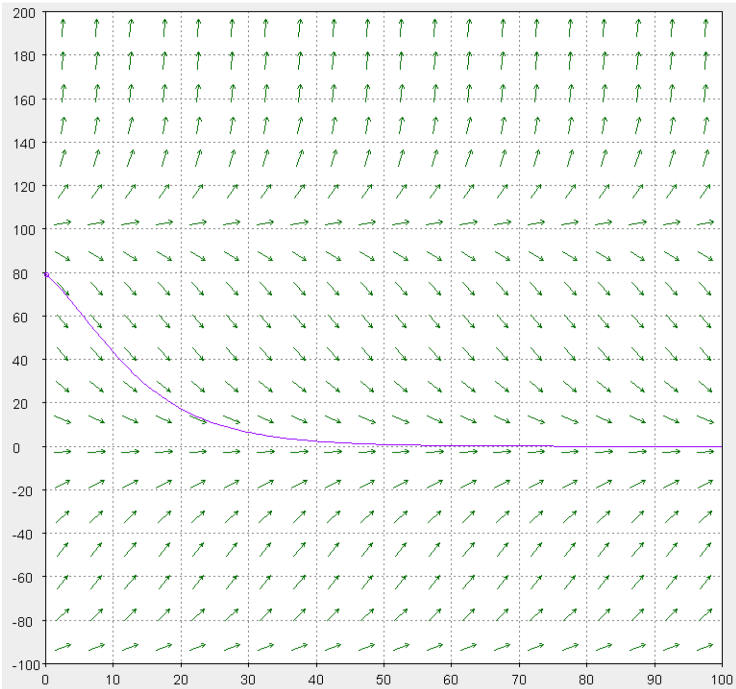
5. Let $P(t)$ represent the population of the Puffy dog breed. Suppose you come up with the following differential equation that models $P(t)$:

$$\frac{dP}{dt} = P\left(P-100\right)\left(P+100\right)/100000$$

Its Slope Field is given by: 

Suppose that the population of the Puffy dog is 80 at time $t = 0$. What is the long term behavior for the population of the Phan fish? Will it keep increasing/decreasing, stabilize to a certain number, or go extinct?

SOLUTION: If $P(0) = 80$, the solution goes through the point $(0, 80)$. Similar as the question above, we can prediction a solution curve like this



Hence by my sketch we can make a guess that

$$\lim_{t\to\infty} P(t) = 0,$$

hence we model that the population of the Puffy dog will go extinct. :(

6. Draw the phase portrait of the DE $\frac{dy}{dx} = y^2(4-y)$, describe the proporties of all possible solutions of the D.E.
SOLUTION: Solve the equation $y^2(4-y) = 0$ to find critical points. We get $y_1 = 0$ and $y_2 = 4$.
• when $y > 4$, $y' = y^2(4-y) < 0$, downward arrow; •when $0 < y < 4$, $y' > 0$, upward arrow; •when $y < 0$, $y' > 0$, upward arrow.

• If it has an initial value $y_0 < 0$, then the solution will be always increasing and negative; • If it has an initial value $0 < y_0 < 4$, then the solution will be always increasing and takes values between 0 and 4only; •If it has an initial value $y_0 > 4$, the the solution will be always decreasing and larger than 4.

7. Find all solutions to the following differential equations by separation of variables.
   (a) $\frac{dT}{dt} = kT$, where $k \neq 0$ is an constant and $T > 0$.
   SOLUTION:

$$\frac{1}{T}dT = kdt$$
$$\int \frac{1}{T}dT = \int kdt$$
$$\ln|T| = kt + C$$

you can either use the implicit solution $\ln |T| = kt + C$ as your answer or use the explicit solution $T = Ce^{kt}$, $C > 0$, as your answer.

(b) $y' = x\sqrt{x^2+1}$.

SOLUTION:

$$\begin{aligned}
\frac{dy}{dx} &= x(x^2+1)^{\frac{1}{2}} \\
1 \cdot dy &= x(x^2+1)^{\frac{1}{2}}\, dx \\
\int 1 \cdot dy &= \int x(x^2+1)^{\frac{1}{2}}\, dx \\
y &= \frac{1}{\frac{1}{2}+1} \cdot \frac{1}{2} \cdot (x^2+1)^{\frac{1}{2}+1} + C \\
y &= \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C
\end{aligned}$$

(c) $dx + e^{x+2y}dy = 0$.

SOLUTION:

$$\begin{aligned}
e^x \cdot e^{2y} dy &= -dx \\
e^{2y} dy &= -e^{-x} dx \\
\int e^{2y} dy &= \int -e^{-x} dx \\
\frac{1}{2}e^{2y} &= e^{-x} + C
\end{aligned}$$

you can use either the implicit function $\frac{1}{2}e^{2y} = e^{-x} + C$ as your answer or the explicit function $y = \frac{1}{2}\ln(2e^{-x} + C)$ as your answer.