Practice - Week 3 - Solution

September 17, 2019

1. Solve the following D.E.s by using the method of 1st-order linear D.E.

   (a) $\frac{dy}{dx} + 4x^3y = x^3$.

   (b) $xy' + y = \sin 2x$.

   SOLUTION:

   - (a) $P(x) = 4x^3$, $f(x) = x^3$,

   $$\int P(x)dx = \int 4x^3dx = x^4 \qquad \Rightarrow \qquad \mu(x) = e^{\int P(s)dx} = e^{x^4},$$

   $$\begin{aligned} \text{solution } y(x) &= \frac{\int \mu(x)f(x)dx + C}{\mu(x)}, \\ &= \frac{\int x^3e^{x^4}dx + C}{e^{x^4}}, \\ &= \frac{\frac{1}{4}e^{x^4} + C}{e^{x^4}}, \\ &= \frac{1}{4} + Ce^{-x^4}. \end{aligned}$$

   (note: u-sub, let $u = x^4, du = 4x^3dx$).

   - (b) Rewrite the D.E.

   $$y' + \frac{1}{x}y = \frac{\sin(2x)}{x} \qquad \Rightarrow \qquad P(x) = \frac{1}{x},\ f(x) = \frac{\sin(2x)}{x},$$

   $$\int P(x)dx = \int \frac{1}{x}dx = \ln|x| \qquad \Rightarrow \qquad \mu(x) = e^{\ln|x|} = x, \text{(by algebra fact)},$$

   $$\begin{aligned} \text{solution } y(x) &= \frac{\int x \cdot \frac{\sin(2x)}{x}dx + C}{x} \\ &= \frac{\int \sin(2x)dx + C}{x} \\ &= \frac{-\frac{1}{2}\cos(2x) + C}{x} \end{aligned}$$

   (note: $u = 2x, du = 2dx$).

2. Solve the following D.E.s by using the method of 1st-order linear D.E. and separation of variables.

   (a) $\frac{dy}{dx} = 5y$.

   (b) $y' + 3x^2y = x^2$.

   SOLUTION:

- (a) 1st-order linear method, $y' - 5y = 0$, $P(x) = -5$, $f(x) = 0$,

$$\mu(x) = e^{\int P(x)dx} = e^{\int -5dx} = e^{-5x},$$

$$\text{solution } y(x) = \frac{\int e^{-5x} \cdot 0dx + C}{e^{-5x}} = \frac{C}{e^{-5x}} = Ce^{5x}.$$

  Separation of variables,

$$\begin{aligned} \frac{1}{y}dy &= 5dx, \\ \ln|y| &= 5x + C, \\ y &= e^{5x+C}, \\ y &= Ce^{5x}. \end{aligned}$$

- (b) 1st-order linear method, $P(x) = 3x^2$, $f(x) = x^2$,

$$\mu(x) = e^{\int P(x)dx} = e^{\int 3x^2dx} = e^{x^3},$$

$$\text{solution } y(x) = \frac{\int e^{x^3} \cdot x^2dx + C}{e^{x^3}} = \frac{\frac{1}{3}e^{x^3} + C}{e^{x^3}} = \frac{1}{3} + Ce^{-x^3}.$$

  Separation of variables,

$$\begin{aligned} \frac{dy}{dx} &= x^2 - 3x^2y, \\ \frac{dy}{dx} &= x^2(1-3y), \\ \frac{1}{1-3y}dy &= x^2, dx, \\ -\frac{1}{3}\ln|1-3y| &= \frac{1}{3}x^3 + C, \text{(implicit solution)} \\ y &= Ce^{-x^3} + \frac{1}{3}, \text{(explicit solution)} \end{aligned}$$

  (note: are you able to get the explicit solution from the implicit solution? please have a try if you are in the honor program).

- Different method leads to the same solution.

3. Solve the following D.E.s.

   (a) $\left(\frac{3}{y^5}\right)\frac{dy}{dt} + \frac{t}{2y^4} + \frac{5}{y^5} = 0.$

   (b) $y' - y = 2te^{2t}$.

   (c) $ty' + 2y = \sin t$, $t > 0$.

   SOLUTION:

   - (a) multiplying $\frac{y^5}{3}$ on both sides

$$\begin{aligned} y' + \frac{t}{6}y + \frac{5}{3} &= 0, \\ y' + \frac{t}{6}y &= -\frac{5}{3}, \end{aligned}$$

$$\int P(t)dt = \int \frac{t}{6}dt = \frac{1}{12}t^2 \qquad \Rightarrow \qquad \mu(t) = e^{\frac{1}{12}t^2},$$

$$\begin{aligned} \text{solution } y(t) &= \frac{\int e^{\frac{1}{12}t^2} \cdot \left(-\frac{5}{3}\right) dt + C}{e^{\frac{1}{12}t^2}}, \\ &= \frac{-\frac{5}{3}\int e^{\frac{1}{12}t^2}dt + C}{e^{\frac{1}{12}t^2}}, \qquad \text{(this will count correct)} \\ &= -\frac{5}{3}e^{-\frac{1}{12}t^2}\int_0^t e^{\frac{1}{12}x^2}dx + Ce^{-\frac{1}{12}t^2}. \qquad \text{(this is a more precise answer).} \end{aligned}$$

(note: the antiderivative of $e^{\frac{1}{12}x^2}$ does not have an explicit expression, so we use the integral $\int e^{\frac{1}{12}x^2}dx$ to denote its antiderivative. Such function(antiderivative) is called an integral defined function).

- (b) $P(t) = -1$, $f(t) = 2te^{2t}$,

$$\mu(t) = e^{\int P(t)dt} = e^{\int -1dt} = e^{-t},$$

$$y(t) = \frac{\int \mu f dt + C}{\mu} = \frac{\int e^{-t} \cdot 2te^{2t}dt + C}{e^{-t}} = e^t(\int 2te^t dt + C) = 2e^t(te^t - e^t) + Ce^t.$$

(note: integration by parts)

- (c) Rewrite

$$y' + \frac{2}{t}y = \frac{\sin t}{t}, \qquad P(t) = \frac{2}{t},\ f(t) = \frac{\sin t}{t},$$

$$\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = t^2,$$

$$y(t) = \frac{\int t^2 \frac{\sin t}{t} dt + C}{t^2} = \frac{\int t \sin t dt + C}{t^2} = \frac{-t\cos t + \sin t + C}{t^2}$$

(last step needs integration by parts, from cal 2).

4. Solve the following initial value problem:

$$ty' + (t+1)\,y = 2te^{-t}, \quad y(1) = a, \quad t > 0.$$

SOLUTION: Solve the D.E. first, rewrite $y' + \frac{t+1}{t}y = 2e^{-t}$, so $P(t) = \frac{t+1}{t}$, $f(t) = 2e^{-t}$,

$$\int P(t)dt = \int \frac{t+1}{t}dt = \int (1 + \frac{1}{t})dt = t + \ln|t|,$$

$$\mu(t) = e^{t+\ln|t|} = e^t e^{\ln|t|} = te^t,$$

$$y(t) = \frac{\int te^t \cdot 2e^{-t}dt + C}{te^t} = \frac{t^2 + C}{te^t},$$

use initial condition,

$$a = y(1) = \frac{1 + C}{1 \cdot e}, \qquad \Rightarrow \qquad C = ae - 1.$$

5. Find the general solutions for the following differential equations. (Determine what formula you want to use first!)

(a) $y' = \dfrac{x^2}{y}$,

(b) $x\frac{dy}{dx} = 2e^x - y + 6x^2$,

(c) $(\cos x - \cos x \sin^2 y)dx + \frac{\cos^2 y}{\sin x}dy = 0$,

(d) $\dfrac{dy}{dx} = \dfrac{x^2 - 3}{3 + 2y}$,

(e) $\frac{dy}{dt} = (y+1)\,(y-2)$.

SOUTION:

- (a) $ydy = x^2dx$, $\frac{1}{2}y^2 = \frac{1}{3}x^3 + C$.
- (b) $y' + \frac{1}{x}y = 6x + \frac{2}{x}e^x$, $P(x) = \frac{1}{x}$, $f(x) = 6x + \frac{2}{x}e^x$,

$$\int P(x)dx = \int \frac{1}{x}dx = \ln|x|, \qquad \Rightarrow \qquad \mu(x) = e^{\ln|x|} = x,$$

$$y(x) = \frac{\int x(6x + \frac{2}{x}e^x)dx + C}{x} = \frac{2x^3 + 2e^x + C}{x}.$$

- (c) use the identity $\sin^2 x + \cos^2 x = 1$,

$$\begin{aligned}
\cos x(1 - \sin^2 y)dx + \frac{\cos^2 y}{\sin x} dy &= 0, \\
\cos x \cos^2 y dx + \frac{\cos^2 y}{\sin x} dy &= 0, \\
\cos x dx &= -\frac{1}{\sin x} dy, \\
\sin x \cos x dx &= -dy, \\
\frac{1}{2}\sin^2 x &= -y + C.
\end{aligned}$$

- (d) separable, implicite solution $\frac{1}{3}x^3 - 3x = 3y + y^2 + C$.
- (e) separable,

$$\begin{aligned}
\frac{1}{(y+1)(y-2)} dy &= dt, \\
\int \frac{1}{(y+1)(y-2)} dy &= \int 1 dt, \\
\int \frac{1}{3}\left(\frac{1}{y-2} - \frac{1}{y+1}\right) dy &= \int 1 dt, \\
\frac{1}{3}(\ln|y-2| - \ln|y+1|) &= t + C, \text{(implicit solution)}
\end{aligned}$$

(note: for students converting honor, can we write an explicit solution? what would it be?)

6. Chanllenge Problem (NOT ON QUIZ OR EXAM). Verify the solution formula for the 1st-order linear D.E., i.e., the following function

$$y(x) = \frac{\int \mu(x) f(x) dx + C}{\mu(x)}, \qquad \mu(x) = e^{\int P(x)dx},$$

is a 1-parameter family of solution to the D.E.

$$y' + P(x)y = f(x).$$

SOLUTION: See Remark in Lecture Notes 1st order linear D.E.

- Thanks everyone for helping me creating a typo free solution!!!