HONOR'S CONVERSION PROBLEMS - 2019 FALL

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1. Laplace Transform

We can easily use induction to prove the formula for the Laplace transform of the $n-$th order derivative (deltail in the previous file).

2. Wronskin

Let us recall the definition of linear dependency of functions:
• We say the n functions $f_1(x)$, $f_2(x)$, ..., $f_n(x)$ are linearly dependent if we can find constants c_1, c_2, ..., c_n, among which at least one of them is **not zero**, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0. \tag{1}$$

To show the dependency of functions, we only need to find a set of appropriate values for $c_1, \ldots, c_n$. However, to show the independency of the functions, we need to show that such not-all-zero constants $c_1, \ldots, c_n$ does not exist. First question is how do we show the independency of the functions? There are infinite numbers, so we can't just say we have tried all the possibilities and we found nothing so it does not exist. Luckily, we can use the Wronskin matrix to test if the functions are dependent or not.
• A little something from linear algebra. We say a (squre) matrix is invertible if its deteminant is not zero. The inverse of a $(n \times n)$ matrix A is denoted as A^{-1}, which is also a $(n \times n)$ matrix such that

$$A \cdot A^{-1} = A^{-1} \cdot A = I_n,$$

where I_n is the identity matrix of dimension $n \times n$.
• For a system of linear equation (not differential equation), we have the following equivalent expression

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{pmatrix} \quad \Leftrightarrow \quad AX = C,$$

when A is invertible, i.e., $detA \neq 0$, then there exists an inverse matrix A^{-1} of A, the linear system of equations can be solved by multiplying the inverse matrix A^{-1} on both sides of the equation:

$$A^{-1} \cdot AX = A^{-1}C, \quad (A^{-1}A)X = A^{-1}C, \quad I_n X = X = A^{-1}C.$$

• Wronskin matrix: Assume there are n functions $f_1(x), f_2(x), \ldots, f_n(x)$, the Wronskin matrix of the functions is defined as

$$W(f_1, \ldots, f_n) = \det \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}_{n \times n}. \tag{2}$$

• The idea of the square matrix is kind of a higher dimension version of numbers. A non zero number a has an inverse $a^{-1} = \frac{1}{a}$ such that $a \cdot \frac{1}{a} = 1 = \frac{1}{a} \cdots a$, here 1 is the dimension one identity (just the number one). For the number zero, the fraction $\frac{1}{0}$ is not well-defined.

• **Problem**: Prove that $f_1, \ldots, f_n$ are linearly independent is equivalent to $W(f_1, \ldots, f_n) = 0$. Similarly, prove that $f_1, \ldots, f_n$ are linearly dependent is equivalent to $W(f_1, \ldots, f_n) \neq 0$.

3. Homogeneous Linear System with Constant Coefficients

3.1. **System of 2 functions.** It is lucky that we don't need too much linear algebra for a systme of 2 functions. The only thing we need is eigenvalue and eigenvectors. Soon you will learn how to use the eigenvalues and eigenvectors to assemble a general solution. But where did the formula come from? Why do we need the eigenvalues and eigenvectors?

The linear system question we deal with is

$$\begin{cases} x_1' = ax_1 + bx_2, \\ x_2' = cx_1 + dx_2, \end{cases} \quad \Leftrightarrow \quad X' = AX, \text{ where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The eigenvalues of A are the roots of the (characteristic polynomial) $\det(A - \lambda I_{2\times 2}) = 0$, let us use λ_1 and λ_2 to denote the two roots of this quaratic equation.

3.1.1. *Distinct Roots.* When $\lambda_1 \neq \lambda_2$.

• A property of eigenvalue, for an eigenvalue λ_1, we know that $\det(A - \lambda_1 I_{2\times 2}) = 0$, it means that the matrix $\begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix}$ is not invertible. Or, equivalently, it means the two row vectors are scalar multiple of each other, i.e., $\frac{a-\lambda_1}{c} = \frac{b}{d-\lambda_1} = k_1$, here k_1 denotes the scalar ratio.

• Let us subtract $\lambda_1 x_1$ and $\lambda_1 x_2$ from the two differential equation respectively, i.e.,

$$\begin{cases} x_1' - \lambda_1 x_1 = (a-\lambda_1)x_1 + bx_2, \\ x_2' - \lambda_1 x_2 = cx_1 + (d-\lambda_1)x_2. \end{cases} \quad \Rightarrow \quad \frac{x_1' - \lambda_1 x_1}{x_2' - \lambda_1 x_2} = k_1 \quad \Rightarrow \quad x_1' - k_1 x_2' = \lambda_1(x_1 - k_1 x_2),$$

we can solve for $x_1 - k_1 x_2$ from the last equation above. Solve for $u_1 = x_1 - k_1 x_2$ by separation of variable.

• Assume the scalar ratio for the second eigenvalue λ_2 is $k_2 = \frac{a-\lambda_2}{c} = \frac{b}{d-\lambda_2}$, use the same method from the previous step to sovle for $x_1 - \lambda_2 x_2$.

• Use your solutions for $x_1 - \lambda_1 x_2$ and $x_1 - \lambda_2 x_2$ to solve for x_1 and x_2.

• Write solutions of x_1 and x_2 in matrix form (it will be in the form of a vertical vector in this case. Compare your general solution in matrix form, try to figure out the relation betwen the scalar ratio k_1, k_2 and the eigenvector from the fomular in class.)

3.1.2. *Repeated Roots.* When $\lambda_1 = \lambda_2$, let us use $\lambda = \lambda_1 = \lambda_2$ for this value.

• For this situation, we need to use the fact that there are two linearly independent eigenvectors associated to the repeated eigenvalue λ. i.e., there are linearly independent vectors $K_1 = \begin{pmatrix} k_{11} \\ k_{12} \end{pmatrix}$ and $K_2 = \begin{pmatrix} k_{21} \\ k_{22} \end{pmatrix}$ such that

$$(A - \lambda I_2)K_1 = 0, \quad \text{and} \quad (A - \lambda I_2)K_2 = 0.$$

For this reason, it is not hard to see that

$$\text{one element in the matrix} \quad A - \lambda I_2 = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \quad \text{has to be zero.}$$

i.e., $a - \lambda = 0$, or $b = 0$, or $c = 0$, or $d - \lambda = 0$. We can use this property to derive the solutions for the system differential equation.

• Another way to view this situation is to use the Jordan normal.

3.2. **Linear System of more than 2 functions.** If you are interested in how to derive the formula in general situation, there is a list of key words to check out. You can follow the hints to do an independent study (research).

(1) For a linear system of n functions: $X' = AX$.
(2) Assume J is the Jordan normal form of A, then there is an invertible matrix P such that $J = P^{-1}AP$, i.e., $PJ = AP$.
(3) Observe the following equation
$$X' = AX = APP^{-1}X = AP(P^{-1}X) = PJ(P^{-1}X), \quad \Rightarrow \quad P^{-1}X' = J(P^{-1}X),$$
do substitution $X_{\text{new}} = P^{-1}X$, then $X'_{\text{new}} = P^{-1}X'$. The orginal linear system $X' = AX$ is equivalent to
$$X'_{\text{new}} = JX_{\text{new}}.$$
(4) The linear system $X'_{\text{new}} = JX_{\text{new}}$ is easy to solve for X_{new} due to the special form of the Jordan normal form J.
(5) The general solution for X can be given by $X = PX_{\text{new}}$.

note: Is there any relation between the matrix P in the above description and the eigenvectors of the matrix A?