Lecture Notes

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Contents

Preface

Due to the fact that the majority of students who are taking MATH2410 FALL 2019 are Engineering majors, Applied math majors or others but pure math major. The class and the lecture notes will focus on how to solve O.D.E.s and the idea behind each formulas. The existence and uniqueness will only be mentioned, let us leave that part to real analysis couse or modern analysis etc.

1 Introduction

1.1 Definitions and Terminologies

Definition 1. An equation is called a **differential equation** if the equation contains any derivative of unknown function(s). We usually use **DE** to denote differential equation.

Example 1.1. $f' = x + f$ is a DE, $f = f(x)$ is the unknown function.

Example 1.2. $\frac{dy}{dx} = 2$ is a DE, $y = y(x)$ is the unknown function.

Remark 1. The following notations are equivalent for the derivatives of function $y(x)$,

$$\begin{aligned} \text{1st order derivative :} \quad & \frac{dy}{dx} = y', \\ \text{2nd order derivative :} \quad & \frac{d^2y}{dx^2} = y'', \\ \cdots\cdots \quad & \cdots \\ \text{n'th order derivative :} \quad & \frac{d^ny}{dx^n} = y^{(n)}. \end{aligned}$$

The left hand notation is called Leibniz notation, the right hand side notation is called prime notation.

Definition 2. The **order** of a differential equation is the order of the highest derivative in the equation.

Example 2.1. Equation

$$y^5 + y' + 3 = y^{-2}$$

is a first order DE since the highest order derivative is given by the term y', which is a first order derivative (other terms in the equation $y^5, 3$ and y^{-2} are either constant or powers of y, not derivative).

Example 2.2. Equation

$$y^{(3)} + y' + 3 = x$$

is a third order DE since the highest order derivative is given by the term $y^{(3)}$, which is a thrid order derivative (the other terms in the equation: y' is a first order derivative, not the highest order derivative; 3 is a constant; x is a term by the independent variable).

Definition 3. For a 1st order differential equation, it is benificial to represent the DE as a **differential form**

$$M(x,y)dx + N(x,y)dy = 0.$$

Example 3.1. Recall the DE in example 1.1

$$y' = x + y,$$

it is equivalent to

$$\frac{dy}{dx} = x + y,$$

multiplying dx to eliminate the fraction to get its differential form

$$dy = (x+y)dx, \quad \Rightarrow \quad (x+y)dx - dy = 0,$$

where $M(x,y) = x + y$ and $N(x,y) = -1$ in the definition.

Example 3.2. The differential equation $y^{(3)} + y' + 3 = x$ can not be expresssed in differential form becuase it is not a 1st order D.E.

Definition 4 (normal form)**.** An explicit expression of the highest order derivative dedrived from the D.E. For example, $y^{(3)} = -y' + x - 3$ is the normal form of the D.E. $y^{(3)} + y' + 3 = x$. And $y' = y^{-2} - y^5 - 3$ is the normal form of the D.E. $y^5 + y' + 3 = y^{-2}$.

So far, we call f or y in the previous examples the unknown function, which is a function depends on x. Considering the unknown function also changes due to different value of x, it is natural to call the unknown function a **dependent variable**, and x an independent variable.

Definition 5 (Linearity)**.** A D.E. is called linear If the it is linear in $\{y, y', y'', \dots, y^{(n)}, \dots\}$, i.e., if the D.E. is in the form

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = g(x).$$

In human language, if every addition term in the D.E. does not contain any of the followings nonlinear functions regarding $y, y', \dots$,

$$\{y^2, y^3, \dots, (y')^2, (y')^3, \dots, (y'')^2, (y'')^3, \dots, (y^{(n)})^2, (y^{(n)})^3, \dots, \sin y, \cos y, \dots\}.$$

Example 5.1. Consider the following equations, determine it is linear or not and explain your reason.

- $(y-x)dx + 4xdy = 0$ — Yes, $(y-x) + 4x\frac{dy}{dx} = 0 \Leftrightarrow 4xy' + y - x = 0$
- $y'' - 2y + y = 0$ — Yes,
- $x^3\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$ — Yes, $x^3y^{(3)} + xy' - 5y = e^x$
- $(1-y)y' + 2y = e^x$ — No, $y' - yy' + 2y = e^x$, the term yy' is not linear
- $\frac{d^2y}{dx^2} + \sin y = 0$ — No, the term $\sin y$ is not linear
- $\frac{d^4y}{dx^4} + y^2 = 0$ — No, the term y^2 is not linear

Definition 6 (Solution of an D.E.)**.** Any funtion that satisfies the D.E. The way to varify solution(s) is to check if the left hand side equals the right hand side of the D.E.

Remark 2. The solution of D.E.s is a FUNCTION, not a number.

Example 6.1. The function $y = \frac{1}{16}x^4$ is a solution of the D.E. $\frac{dy}{dx} = xy^{\frac{1}{2}}$ on the interval $(-\infty, \infty)$.

Solution:

$$L.H.S. = y' = (\frac{1}{16}x^4)' = \frac{1}{4}x^3,$$

$$R.H.S. = xy^{\frac{1}{2}} = x\left(\frac{1}{16}x^4\right)^{\frac{1}{2}} = \frac{1}{4}x^3,$$

the solution is verified since L.H.S.=R.H.S.

Example 6.2. The function $y = xe^x$ is a solution of the D.E. $y'' - 2y' + y = 0$ on the interval $(-\infty, \infty)$.

You may notice that the interval which the solution is defined on is specified, it is because the solution is a function, and functions are not always well defined everywhere. For example, the function $y = \sqrt{1 - x^2}$ is well defined on intervel $(-1, 1)$, meanwhile, it is a solution of the D.E. $y' = -\frac{x}{y}$. We call this interval $(-1, 1)$ the domain of solution.

Definition 7 (Implicit Solution)**.** A relation $G(x, y) = 0$ between x and y is said to be an implicit solution of a D.E. if it satisfies the D.E. In human language, sometime the solution of a D.E. can be given as an equation of x, y instead of a function y of x.

The human language version of the definition is still pretty confusing. Let us look at the following example.

Example 7.1. Consider the equation $x^2 + y^2 = 1$ is a relaton between x and y. Solve for y, we have $y = \pm\sqrt{1 - x^2}$, do derivative

$$y' = \mp x(1 - x^2)^{-\frac{1}{2}} = \frac{-x}{\pm\sqrt{1 - x^2}} = -\frac{x}{y}.$$

The the solution of $y' = -\frac{x}{y}$ can be expressed as $x^2 + y^2 = 1$, or $y = \pm\sqrt{1 - x^2}$. The differential form expression of D.E.s gives benifits in verifying implicit solutions. Do derivative (differential operator) to the equation implicit solution

$$\begin{aligned} d(x^2 + y^2) &= d(1) \\ 2xdx + 2ydy &= 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{x}{y}. \end{aligned}$$

With the help the concept of implicit solution, the solution can be given as one 'pretty' equation $x^2 + y^2 = 1$ instead of the 'annoying' one $y = \pm\sqrt{1 - x^2}$.

Example 7.2. The equation $-2x^2y + y^2 = 1$ is an implicit solution of the D.E.

$$2xydx + (x^2 - y)dy = 0 \quad \text{(notice that this D.E. is in differential form).}$$

It is not easy to get an explicit expression of y in x from the equation $-2x^2y + y^2 = 1$, that's why we introduce the implicit solution. To verify it, differentiate the equation

$$\begin{aligned} d(-2x^2y + y^2) &= d(1) \\ (-2 \cdot 2x \cdot dx \cdot y - 2x^2 \cdot dy) + 2y \cdot dy &= 0 \\ -2[2xydx - (x^2 - y)dy] &= 0 \\ 2xydx + (x^2 - y)dy &= 0, \quad \text{this is exactly the D.E. in the question.} \end{aligned}$$

So the equation satisfies the D.E., it is a solution.

Definition 8 (Families of Solution)**.** Let us look at an example first, a simple D.E. $y' = 2x$, we can find its solution by integration, i.e., $y = \int y' = \int 2x = x^2 + C$, recall the arbitrary constant C is called a parameter in calculus. This example shows that the solution function of a D.E. could be more one, for this situation, we call solutions $y = x^2 + C$ with one arbitrary constant a one-parameter family of solutions. **An n-th order D.E. has an n-parameter family of solutions.**

Example 8.1. Verify $y = Cx^4$ is a one-parameter family of solutions of the first-order equation $xy' - 4y = 0$.

Example 8.2. Verify $y = C_1e^x + C_2xe^x$ is a two-parameter family of solutions of the second-order equation $y'' - 2y' + y = 0$.

Solution:

$$\begin{aligned} y' &= C_1e^x + C_2(e^x + xe^x) = (C_1 + C_2)e^x + C_2xe^x \\ y'' = (y')' &= [(C_1 + C_2)e^x + C_2xe^x]' \\ &= [(C_1 + C_2)e^x + C_2(e^x + xe^x)] \\ &= (C_1 + 2C_2)e^x + C_2xe^x, \end{aligned}$$

therefore

$$\begin{aligned} L.H.S &= y'' - 2y' + y \\ &= (C_1 + 2C_2)e^x + C_2xe^x - 2[(C_1 + C_2)e^x + C_2xe^x] + C_1e^x + C_2xe^x \\ &= 0 = R.H.S. \end{aligned}$$

Example 8.3. Could $y = C_1x + C_2x^2$ be a two-parameter family of solutions of the D.E. $y' = 1 + 2x$? Explain your reason.

Definition 9. Initial-Value Problems (I.V.P.) is seeking solution(s) for a D.E. with given initial condition(s). For example, the following problem is an I.V.P.,

$$\begin{cases} y' = 2x, & \text{the D.E.} \\ y(0) = 3, & \textbf{initial condition} \end{cases}$$

The D.E. $y' = 2x$ has a one-parameter family of solutions $y = x^2 + C$, in order make the solution to satisfy the initial condition, let us take value $x = 0$ to the family of solution,

$$y(x = 0) = 0^2 + C = 3, \quad \Rightarrow \quad C = 3.$$

Therefore, when the parameter takes value $C = 3$, the solution $y = x^2 + 3$ is a solution to the I.V.P.

From the above example we can roughly say that the initial condition determines values of the parameters in the family solutions of the D.E.

Example 9.1 (second order I.V.P.)**.** Given that the D.E. $y'' + 16y = 0$ has a two-parameter family of solutions $y = C_1 \cos(4x) + C_2 \sin(4x)$. Determine the values of the parameters so that it solves the following I.V.P.

$$\begin{cases} y'' + 16y = 0, \\ y(\frac{\pi}{16}) = -2, \\ y'(\frac{\pi}{16}) = 1. \end{cases}$$

Solution: the initial conditions are given when $x = \frac{\pi}{16}$. From the first initial condition,

$$\begin{aligned} -2 = y(x = \frac{\pi}{16}) &= C_1 \cos(4 \cdot \frac{\pi}{16}) + C_2 \sin(4 \cdot \frac{\pi}{16}) \\ &= C_1 \cos(\frac{\pi}{4}) + C_2 \sin(\frac{\pi}{4}) \\ \Rightarrow \qquad -2 &= \frac{\sqrt{2}}{2} C_1 + \frac{\sqrt{2}}{2} C_2, \end{aligned}$$

From the second initial codition, it is a value for y', calculate y' first,

$$y' = (C_1 \cos(4x) + C_2 \sin(4x))' = -4 \sin(4x) + 4 \cos(4x),$$

$$\begin{aligned} 1 = y'(x = \frac{\pi}{16}) &= -4C_1 \sin(4 \cdot \frac{\pi}{16}) + 4C_2 \cos(4 \cdot \frac{\pi}{16}) \\ &= -4C_1 \sin(\frac{\pi}{4}) + 4C_2 \cos(\frac{\pi}{4}) \\ \Rightarrow \qquad 1 &= -2\sqrt{2}C_1 + 2\sqrt{2}C_2, \end{aligned}$$

therefore, we have a set of equations

$$\begin{cases} \frac{\sqrt{2}}{2} C_1 + \frac{\sqrt{2}}{2} C_2 = -2, & \Rightarrow C_2 = -2\sqrt{2} - C_1, \quad \text{plug in 2nd eq.} \\ -2\sqrt{2}C_1 + 2\sqrt{2}C_2 = 1. & -2\sqrt{2}C_1 + 2\sqrt{2}(-2\sqrt{2} - C_1) = 1, \end{cases}$$

so

$$C_1 = \frac{1 + 2\sqrt{2} \cdot 2\sqrt{2}}{-2\sqrt{2} - 2\sqrt{2}} = -\frac{9\sqrt{2}}{8},$$

$$C_2 = -2\sqrt{2} - (-\frac{9\sqrt{2}}{8}) = -\frac{7}{8}\sqrt{2},$$

the solution of the I.V.P. is

$$y(x) = -\frac{9\sqrt{2}}{8}\cos(4x) - \frac{7}{8}\sqrt{2}\sin(4x).$$

Remark 3. The solution to an I.V.P. may not be unique, for exmple,

$$\begin{cases} y' = xy^{\frac{1}{2}}, \\ y(0) = 0. \end{cases}$$

has two solutions. A trivial one $y(x) = 0$, and $y = \frac{1}{16}x^4$.

1.2 Simple Math Modeling

Definition 10 (Rate of Change)**.** Assume we use function $f(t)$ to denote the quantity f which depends on time t, then the rate at which hte quantity f changes at time t is $\frac{df}{dt}$, i.e., the derivative of $f(t)$ at time t.

Example 10.1. I use the function $M(t)$ to denote the amount of money I possess at Foxwood at time t in minutes. I went there with 10 bucks in my pocket, it means $M(0) = 10$. Assume I make 2 cents (=0.02 dollar) every second, it means $\frac{dM}{dt} = 0.02$ (i.e. $M'(t) = 0.2$). More precisely, no matter when, the amount of money I make is always 0.2 dollar. For example, 600 seconds(=10 minutes) later, I still make 2 more cents from the 600th second to the 601st second, i.e., $M'(600) = 0.02$.

Definition 11 (Proportional)**.** An quantity A is **proportional** to another quantity B means the ratio between them is a constant. Normally the constant is unknown, and we denote it as k. We express it as

$$A \propto B \qquad \Leftrightarrow \qquad A = kB.$$

The constant k is called **the proportional constant**.

Example 11.1. Let us use the function $P(t)$ to denote the population of the dog breed Puffy at time t, then the derivative $\frac{dP}{dt}$ tells us the rate at which the population of Puffy breed at time t (in decades). For example, the following shows the relation between real life informations and math model informations

there are two Puffy dogs at the beginning $\quad \Rightarrow \quad P(0) = 2,$

the rate at which the population of Puffy breed is **proportional** to the total population means

$$\underset{\sim}{\frac{dP}{dt}} \propto \underline{P}.$$

Example 11.2. Suppose that $dP/dt = 0.12P(t)$ represents a mathematical model for the growth of a certain cell culture, where $P(t)$ is the size of the culture (measured in millions of cells) at time $t > 0$ (measured in hours). How fast is the culture growing at the time t when the size of the culture reaches 2 million cells?
Solution: It is the same as asking what is the rate of change of the population at the time t when the population $P(t) = 2 \times 10^6$. We don't need to figure out the exact time t to get the rate of change at that moment. Becuase when $P(t) = 2 \times 10^6$, the rate of grows satisfies $P'(t) = 0.12P(t) = 0.12 \times 2 \times 10^6 = 240000$.

Definition 12 (Newton's Law of Cooling and Warming)**.** Let $T(t)$ be the temperature of an object at time t, and let T_s be the temperature of the surrounding enviroment (normally a constant). The rate at which the temperature of the object change is proportional to eh difference between the temperature of the object and the temperature of the surrounding enviroment. In math modeling,

$$\frac{dT}{dt} \propto (T - T_s) \qquad \Leftrightarrow \qquad \frac{dT}{dt}(t) = -k(T(t) - T_s), \quad k > 0.$$

The proportional constnat k is often called the cooling/warming constant and we define $k > 0$ in our class.

Remark 4. Why do we need the negative sign in the formula? When it is the case of warming, the temperature $T(t)$ increases, so the derivative $T'(t)$ is positive. We know that $T'(t)$ is porpotional to $T(t) - T_s$. In the situation of warming, the surrounding is hotter than the object, which implies that $(T(t) - T_s) < 0$. To make the sign of both sides of the equation matches, we need to put $-$ in front of $k(T(t) - T_s)$, i.e.,

$$T'(t) = -k(T(t) - T_s).$$

How about the case of cooling? Why do we still put $-$ on the R.H.S (we covered in class)?

Example 12.1. A cake with temperature 350^oF is removed from an oven to be placed in a room with temperature 70^oF. Newton's Law of cooling tells us that the rate of change of the temperature of the cake is proportional to the temperature difference between the cake and the room. The proportionality constant is $k = 0.19$. Set up a math model by using differential equation.
Solution: The room is the surrouding enviroment, $T_s = 70$. The following I.V.P. decribes the situation

$$\begin{cases} T(0) = 350, \\ T'(t) = -0.19(T(t) - 70). \end{cases}$$

2 First-Order Differential Equation

2.1 Direction Field and Phase Protrait

For a first order D.E., we normally can express the the first order derivative $y'(x)$ explicitly. For example, we can write

$$xy' + y^2 = e^x \qquad \text{as} \qquad y' = \frac{e^x - y^2}{x}.$$

Consider the special meaning of 1-order derivative, which is that it represents the slope of the solution function $y(x)$ at x. This proporty can tell us informations about the solution of an D.E. without solving it.

Definition 13 (Direction Field)**.** The **Direction Field** of a first order D.E.

$$y' = g(x, y)$$

is a geometric interpretion of the solution by showing the slope of the tangent to the graph of the solution.

Example 13.1. The direction field of $y' = 2$ is

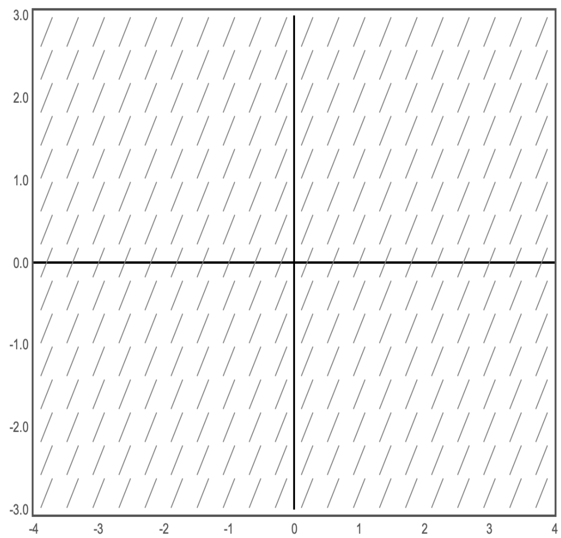

Figure 1: direction field of $y' = 2$

Example 13.2. The direction field of $\frac{dy}{dx} = -\frac{x}{y}$,

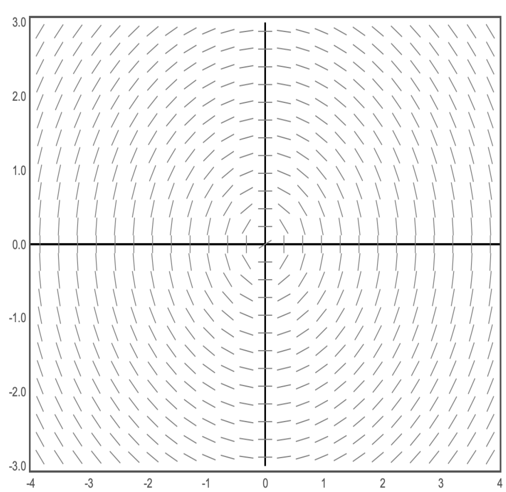

Figure 2: direction field of $y' = -\frac{x}{y}$

Note. *With the concept of direction field, the following special type of D.E.*

$$y' = f(y)$$

deserves special attention, where the derivative depends on the value of the dependent variable y only.
***Definition**. An D.E. in which the independent variable does not appear exlicitly is said to be **autonomous**. **Definition**. For an autonomous D.E. $y' = f(y)$, the values for y such that $f(y) = 0$ are called **critical points** or **equilibrium solutions**. **Example**. $y' = y(1-y)$ is an autonomous D.E., $y = 1$ and $y = 0$ are its critical points. **Example**. $y' = x+y$ is not an autonomous D.E., because the independent variable x showed up, therefore it doesn't have any equilibrium solution. **Example**. $y' = 1+y^2$ is an autonomous D.E., but it does not have equilibrium solution since $y^2+1 = 0$ does not have a solution for y. **Phase Portait**. For an autonomous D.E., draw a $y-$axis that is vertical and mark all the critical points on the axis. Exam the sign of y' on each interval that divideded by the critical points, if it is positive, draw an upward arrow on corresponding inverval; if it is negative, draw a downward arrow. **prediction of solution** From the direction field of solutions of the D.E., we can predict how the solution goes when $x \to \infty$. **Example**. Consider the D.E. $dP/dt = P(aP-b)$, where a, b are positive constnats. Discuss what happens to the population P as time t increases. If the initial condition $P(0) > b/a$, then $P(t) \to$ ____ as t increases; if the initial condition $0 < P(0) < b/a$, then $P(t) \to$ ____ as t increases.*

2.2 Separable D.E.

Definition 14. For a 1st-order D.E., if it can be expressed in the following form

$$\frac{dy}{dx} = g(x)h(y),$$

then we call it is **Separable**.

Method 14.1. In short: put everything about x on one side, and everything about y on the other side. Then integrate both sides.

Assume the separable D.E. we want to solve is

$$\frac{dy}{dx} = g(x)h(y).$$

• put everything about x on one side, and everything about y on the other side:

$$\frac{1}{h(y)}dy = g(x)dx.$$

• integrate both sides:

$$\int frac1h(y)dy = \int g(x)dx.$$

Remark 5. The solution we get will be implicit in general.

Example 14.1. Solve $\frac{dy}{dx} = -\frac{x}{y}$.

SOLUTION:

$$\begin{aligned} ydy &= -xdx, && \text{when we separate variables,} \\ \int ydy &= \int -xdx, && \text{it is better if we put } dy \text{ on the left,} \\ \frac{1}{2}y^2 + C_1 &= \frac{1}{2}x^2 + C_2, && \text{and } dx \text{ on the right.} \\ x^2 + y^2 &= C. \end{aligned}$$

Example 14.2. Solve $ydx + (x+1)dy = 0$.

SOLUTION:

$$(x+1)dy = -ydx,$$
$$\frac{1}{y}dy = -\frac{1}{x+1}dx,$$
$$\int \frac{1}{y}dy = \int -\frac{1}{x+1}dx,$$

- $\ln|y| = -\ln|x+1| + C.$
- $y = Ce^{-\ln|x+1|}.$
- $y = \frac{C}{x+1}.$

All three functions with the bullet can be your answer (I try to minimize the algebra part).

Example 14.3. Solve

$$\frac{e^{2y}}{\cos x}y' = e^y \sin x.$$

SOLUTION:

$$\begin{aligned} \frac{e^{2y}}{\cos x}\frac{dy}{dx} &= e^y \sin x, && \text{If you are familiar with double angle} \\ \frac{e^{2y}}{e^y}dy &= \sin x \cdot \cos x dx, && \text{identity } \sin(2x) = 2\sin x \cos x, \\ \int e^y dy &= \int \sin x \cdot \cos x dx, && \text{feel free to use it.} \\ e^y &= \frac{1}{2}(\sin x)^2 + C. \end{aligned}$$

2.3 1st-order linear D.E.

Definition 15. For a 1st-order linear D.E. in the form of

$$y' + P(x)y = f(x),$$

its general solution is

$$y(x) = \frac{\int \mu(x) f(x) dx + C}{\mu(x)},$$

where $\mu(x) = e^{\int P(x)dx}$.

Method 15.1. For a 1st order differential equation

$$a_1(x)y' + a_0(x)y = g(x),$$

we can solve for its general solution by following steps,

1. Rewrite the 1st-oreder linear D.E. as the following form

$$y' + P(x)y = f(x).$$

2. Determine the corresponding $P(x)$ and $f(x)$.
3. Calculate $\int P(x)dx$ to get the integrating factor

$$\mu(x) = e^{\int P(x)dx}.$$

4. The general solution is

$$y(x) = \frac{\int \mu(x) f(x) dx + C}{\mu(x)}.$$

Remark 6. DO NOT add the extra constant to the integrations in this Method. The arbitrary constant is added.

Example 15.1. To use the method to solve the 1st-order linear D.E.

$$x^2 y' - x^8 = 3xy,$$

(**Step 1**)we need rewrtie it in the form of $y' + P(x)y = f(x)$,

every term with y on the left, rest on the other side: $x^2y' - 3xy = x^8$,

devide by x^2 such that the coefficient of y' is 1: $y' - \frac{3}{x}y = x^5$.

(**Step 2**)Now we are able to determine $P(x) = -\frac{3}{x}$, and $f(x) = x^5$ because

$$y' + \left(-\frac{3}{x}\right) y = x^5,$$

attention, the negative sign is crucial!!! (**Step 3**) is

$$\int P(x)dx = -\int \frac{3}{x} dx = -3 \ln |x|,$$

so the integrating factor

$$\mu(x) = e^{\int P(x)dx} = e^{-3\ln|x|} = (e^{\ln|x|})^{-3} = x^{-3},$$

attention, $e^{\ln|x|} = x$ is an important algebra fact in our class, and so as $e^{ab} = (e^a)^b$, please bring your memory back (since you have met all the prerequisites fto register Math 2410). (**Step 4**)The general solution is

$$y(x) = \frac{\int x^{-3} \cdot x^5 dx + C}{x^{-3}} = \frac{\frac{1}{3}x^3 + C}{x^{-3}},$$

attention, if you leave your answer as the above, **make sure** the arbitrary constant C is included in the numerator!!! If you would like to simplify it, the solution is

$$y(x) = \frac{1}{3}x^3 + Cx^3.$$

Note. *Before we step into how to get the solution formula, let us to convince ourselves that the formula is correct. Recall that the verification of a function being a solution of a D.E. is to check if the L.H.S. matches the R.H.S. when we substitue y with the function.*

- *To verify $y(x) = \frac{\int \mu(x)f(x)dx+C}{\mu(x)}$ is a solution of $y' + P(x)y = f(x)$. Let us check the L.H.S.*

$$\begin{aligned}
y' + P(x)y &= \left(\frac{\int \mu(x)f(x)dx + C}{\mu(x)}\right)' + P(x)\frac{\int \mu(x)f(x)dx + C}{\mu(x)} \\
&= \frac{(\int \mu f dx + C)'\mu - (\int \mu f dx + C)\mu'}{\mu^2} + P(x)\frac{\int \mu f dx + C}{\mu} \\
&= \frac{\mu f \cdot \mu - (\int \mu f dx + C)\mu'}{\mu^2} + P(x)\frac{\int \mu f dx + C}{\mu} \\
&= f - \frac{\int \mu f dx + C}{\mu} \cdot \frac{\mu'}{\mu} + P(x)\frac{\int \mu f dx + C}{\mu} \\
&= f + \left(P(x) - \frac{\mu'}{\mu}\right)\frac{\int \mu f dx + C}{\mu} \\
&= f + \left(P(x) - \frac{(e^{\int P(x)dx})'}{e^{\int P(x)dx}}\right)\frac{\int \mu f dx + C}{\mu} \\
&= f + \left(P(x) - \frac{e^{\int P(x)dx}(\int P(x)dx)'}{e^{\int P(x)dx}}\right)\frac{\int \mu f dx + C}{\mu} \\
&= f + (P(x) - P(x))\frac{\int \mu f dx + C}{\mu} \\
&= f = R.H.S.
\end{aligned}$$

Note. *Now, if you wonder how did people come up with the solution, here is a natural explaination. Our goal is to find solution of a first order linear differential equation*

$$a_1(x)y' + a_0(x)y = g(x).$$

Let us start with gussing the form of its implicit solution. If the implicit function $xy^2 + y + 2x = 0$ is a solution, differentiate the implicit function with respect to x, we have $y^2 + x \cdot 2yy' + y' + 2 = 0$ where we consider $y = y(x)$ as a function of x. The equation we get from differentiating the implicit function is a first-order D.E., however it is not linear because of the term $(xy^2)' = y^2 + 2xyy'$. If an implicit function includes terms of special functions of y, the differentiation of the implicit function can't lead to a linear D.E. as well. For example, differentiation of $e^y + x^3 = 0$ is $y'e^y + 3x^2 = 0$, which is not a linear D.E.. Let us conclude our assumption of the form of the solution to a 1st order linear D.E. $y' + P(x)y = f(x)$ as

$$u(x)y(x) + v(x) = 0, \tag{1}$$

which is an implicit function only has linear y-terms. By our assumption, equation (1) is a solution of $y' + P(x)y = f(x)$, the differential equation $u'y + uy' + v' = 0$, which we get from differentiating the implicit solution, should be equivalent to $y' + P(x)y = f(x)$. More precisely, if we rewrite $u'y + uy' + v' = 0$ as

$$y' + \frac{u'}{u}y = -\frac{v'}{u},$$

this D.E. should be exactly the D.E. $y' + P(x)y = f(x)$, i.e.,

$$\frac{u'(x)}{u(x)} = P(x) \quad \textit{and} \quad -\frac{v'(x)}{u(x)} = f(x).$$

Therefore we have

$$\int \frac{u'(x)}{u(x)}dx = \int P(x)dx \quad \Rightarrow \quad \ln|u(x)| = \int P(x)dx + C_1, \quad \Rightarrow \quad u(x) = \tilde{C}_1 e^{\int P(x)dx}.$$

$$v'(x) = -f(x)u(x) = -\tilde{C}_1 e^{\int P(x)dx} f(x), \quad \Rightarrow \quad v(x) = -\tilde{C}_1 \int e^{\int P(x)dx} f(x)dx + C_2,$$

so an explicit solution can be dedrived from the implicit solution

$$\begin{aligned} uy + v &= 0 \\ y &= -\frac{v}{u} \\ &= -\frac{-\tilde{C}_1 \int e^{\int P(x)dx} f(x)dx + C_2}{\tilde{C}_1 e^{\int P(x)dx}} \\ y &= \frac{\int e^{\int P(x)dx} f(x)dx + C}{e^{\int P(x)dx}} \end{aligned}$$

2.4 Exact Differential Equation

Definition 16. A 1st order differential equation expressed in differential form

$$M(x, y)dx + N(x, y)dy = 0$$

is called **exact** if

$$\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y).$$

Remark 7. Sometime we use $M_y = N_x$ to denote the partial derivatives.

Example 16.1. The differential equation $(2x - 5y)dx + (-5x + 3y^2)dy = 0$ is exact. Because

$$M_y = (2x - 5y)_y = -5 \quad \text{and} \quad N_x = (-5x + 3y^2)_x = -5.$$

Method 16.1. To solve an exact differential equation,

1. Determine functions M and N, find the anti-derivative $f(x,y)$ of $M(x,y)$ with respect to x,

$$f(x,y) = \int M(x,y)dx + h(y),$$

2. Do partial derivative of $f(x,y)$ with respect to y and set the equation

$$\frac{\partial f}{\partial y}(x,y) = N(x,y)$$

   to solve for $h(y)$.

3. The implicit function

$$f(x,y) = \int M(x,y)dx + h(y) = 0$$

   is an implicit solution of the exact D.E.

Example 16.2. In the previous example, $M(x,y) = 2x - 5y$ and $N(x,y) = -5x + 3y^2$. (**Step 1**) Assume the solution is

$$f(x,y) = \int M dx + h(y) = \int (2x - 5y)dx + h(y) = x^2 - 5xy + h(y),$$

(**Step 2**) we want $f_y = N$, which means

$$(x^2 - 5xy + h(y))_y = -5x + 3y^2 \quad \Rightarrow \quad h'(y) = 3y^2 \quad \Rightarrow \quad h(y) = y^3.$$

(**Step 3**) so the solution is

$$f(x,y) = x^2 - 5xy + y^3.$$

Remark 8. TRICKS

1. Merge anti-derivatives.
2. Integration is always with respect to the dx that is multiplied right after.

3. Compare to **potential function** in multivariable calculues (which you probably learnt it with 'Fundamental Theorem of Line Integral', an associated concept is 'conservatvie vector field').

Note. *Question: Why do we seperate this type of D.E.s? What's the connection with conservative vector field? Let us start with the solution function of a D.E., in general, the solution can always be expressed as an implicit function*

$$f(x, y) = C.$$

If we apply implicit differentiation to it, we will get a differential equation in differential form

$$f_x(x, y)dx + f_y(x, y)dy = 0.$$

An the other hand, if we start with a differential equation $M(x, y)dx+N(x, y)dy = 0$*, and* $M(x, y)$ *is the partial derivative of some function* f *with respect to* x*, meanwhile* $N(x, y)$ *is the partial derivative of the same function* f *with respect to* y*. Then the implicit function* $f(x, y) = C$ *is the solution of* $Mdx + Ndy = 0$*. In summary, for a differential equation* $M(x, y)dx + N(x, y)dy = 0$*, if there exists a function* $f(x, y)$ *such that*

$$f_x(x, y) = M(x, y) \quad \textit{and} \quad f_y(x, y) = N(x, y),$$

then $f(x, y) = C$ *is ths general solution. To check if the function* $f(x, y)$ *exists, we only need to check the following is true*

$$M_y(x, y) = N_x(x, y) \quad \textit{due to the fact} \quad f_{xy} = f_{yx}.$$

Because this type is special, we name them Exact D.E.s (the function $f(x, y)$ *is called the potential function of the conservative vector field* $< M(x, y), N(x, y) >$*).*

Remark 9. Revisit the solution method, in the first step, the reason we asssume the solution to be

$$f(x, y) = \int M(x, y)dx + h(y)$$

is because of

$$f_x(x, y) = (\int M(x, y)dx + h(y))_x = (\int M(x, y)dx)_x + h(y)_x = M(x, y),$$

where $(h(y))_x = 0$ due to $h(y)$ is a single variable function of y.

2.5 Three Substitution Methods for Certain D.E.s

Method 16.2 (Reducible D.E.)**.** If a differential equation is of the following form

$$\frac{dy}{dx} = f(Ax + By + C),$$

we can use the substitution $u = Ax + By + C$ to make the D.E. become to a separable differential equation of u and x.

Example 16.3. To solve the differential equation

$$\frac{dy}{dx} = \frac{1-x-y}{x+y},$$

we notice that the right hand side is $\frac{1-(x+y)}{x+y}$, if we let $u = x+y$, then the right hand side becomes to $\frac{1-u}{u}$, so it is a good choice for u. Let $u = x+y$, so $du = (x+y)_x dx + (x+y)_y dy = dx + dy \quad \Rightarrow \quad dy = du - dx$, the differential equation becomes to

$$\frac{du-dx}{dx} = \frac{1-u}{u} \quad \Rightarrow \quad \frac{du}{dx} - 1 = \frac{1}{u} - 1 \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{u},$$

$$udu = dx \quad \Rightarrow \quad \frac{1}{2}u^2 = x + C \quad \Rightarrow \quad \frac{1}{2}(x+y)^2 = x + C.$$

Example 16.4. Solve $\frac{dy}{dx} = \frac{1}{\cos(x+y)} - 1$, $u = x+y$, $du = dx+dy$, $dy = du-dx$,

$$\frac{du-dx}{dx} = \frac{1}{\cos u} - 1 \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{\cos u} \quad \Rightarrow \quad \cos u du = dx,$$

$$\Rightarrow \quad \sin u = x + C \quad \Rightarrow \quad \sin(x+y) = x + C.$$

Definition 17. A function $h(x,y)$ is **homogeneous** if

$$h(tx,ty) = t^\alpha h(x,y) \quad \text{for some constant} \quad \alpha.$$

The constant α is called the degree of the homogeneous function.

Definition 18. A 1st order D.E. in differential form

$$M(x,y)dx + N(x,y)dy = 0$$

is a homogeneous differential equation if the functions $M(x,y)$ and $N(x,y)$ are homogeneous of the same degree.

Example 18.1. The functions $h(x,y) = x^2y + xy^2$ is homogeneous because $h(tx,ty) = (tx)^2(ty) + tx(ty)^2 = t^3(x^2y + xy^2) = t^3h(x,y)$, it is homogeneous of degree 3.

Example 18.2. $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$ is homogeneous, because it is equivalent to

$$-ydx + (x + \sqrt{xy})dy = 0 \qquad M(x,y) = y, \quad N(x,y) = x + \sqrt{xy}.$$

Check if they are homogeneous and detemine their degree

$$M(tx,ty) = -ty = t \cdot (-y) = t^1 M(x,y),$$

$M(x,y)$ is homogeneous of degree 1. For N,

$$N(tx,ty) = tx + \sqrt{txty} = tx + \sqrt{t^2xy} = tx + t\sqrt{xy} = t \cdot (x + \sqrt{xy}) = t^1 N(x,y),$$

N is homogeneous of degree 1. $M(x,y)$ and $N(x,y)$ are both homogeneous of degree 1 so the D.E. is homogeneous.

Remark 10. TRICK: As long as each term in the function share the same 'power', the function is homogeneous.

Method 18.1 (Homogeneous)**.** If a differential equation is homogeneous, we use the substitution $y = ux$ to turn the differential equation to a separable D.E. of u and x (or use the substitution $x = vy$ to make it to a separable D.E. of v and y).

Example 18.3. To solve the homogeneous differential equation $-ydx + (x + \sqrt{xy})dy = 0$, let us use the substitution $y = ux$. We need to substitute dy, so $dy = (ux)_x dx + (ux)_u du = udx + xdu$, the differential equation

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y}{x+\sqrt{xy}} \qquad \text{express } y' \text{ explicitly first,} \\
\frac{udx + xdu}{dx} &= \frac{ux}{x+\sqrt{x\cdot ux}} \\
u + x\frac{du}{dx} &= \frac{x\cdot u}{x(1+\sqrt{u})} \\
x\frac{du}{dx} &= \frac{u}{1+\sqrt{u}} - u \\
x\frac{du}{dx} &= \frac{-u\sqrt{u}}{1+\sqrt{u}} \\
(1+u^{\frac{1}{2}})u^{-\frac{3}{2}}du &= -\frac{1}{x}dx \\
\int (u^{-\frac{3}{2}} + u^{-1})du &= \int \frac{1}{x}dx \\
-2u^{-\frac{1}{2}} + \ln|u| &= \ln|x| + C,
\end{aligned}$$

the solution is

$$-2\sqrt{\frac{x}{y}} + \ln|\frac{y}{x}| = \ln|x| + C.$$

2.6 Euler's Method - A Numerical Approach

Method 18.2. For a first order differential equation

$$\frac{dy}{dx} = f(x,y) \qquad \text{with a given initial value} \quad y(x_0) = y_0,$$

we can use the following recursive formula to numerically approximation the value of its solution $y(a)$ when $x = a$,

1. choose a step size h (reasonably small);
2. determine the number of the recursive approximation by $n = \frac{a-x_0}{h}$;
3. repeat the formula from $m = 1$ until $m = n - 1$,

$$y_{m+1} = y_m + h\cdot f(x_m, y_m), \qquad x_m = x_0 + mh.$$

Note. *The idea of Euler's method is to use tangent line to approximate the value. Compare to*

- *Linearization in Cal 1 - tangent line;*
- *Linearization in Multi-Cal - tangent plane.*

Example 18.4. Consider the initial value problem

$$y' = 0.1\sqrt{y} + 0.4x^2, \qquad y(2) = 4,$$

we want to use Euler's method to obtain an approximation of $y(2.5)$ by using step size $h = 0.25$.
In this case, the value we want to approximate is $y(2.5)$, which means $a = 2.5$ in the method; the initial value condition is $y(2) = 4$, which means $x_0 = 2$ and $y_0 = 4$. The numer of steps we will calculate is

$$n = \frac{a - x_0}{h} = \frac{2.5 - 2}{0.25} = 2,$$

which means y_2 will give the approximation of $y(2.5)$. Notice that $f(x, y) = 0.1\sqrt{y} + 0.4x^2$,
when $m = 0$,

$$y_1 = y_0 + hf(x_0, y_0) = 4 + 0.25 \times (0.1 \times \sqrt{4} + 0.4 \times 2^2) = 4.45, \quad x_1 = x_0 + h = 2.25,$$

when $m = 1$,

$$y_2 = y_1 + hf(x_1, y_1) = 4.45 + 0.25 \times (0.1 \times \sqrt{4.45} + 0.4 \times 2.25^2) = 5.0089,$$

So $y_2 = 5.0089$ is the numerical approximation for $y(2.5)$.

Remark 11. See Week 5 Practice (solution) for more examples.

Note. ***Bernolli's Equation:*** *a 1st order D.E. is called Bernolli's equation if it can be expressed in the following form*

$$y' + P(x)y = f(x)y^n.$$

This differential equation is not linear. If we try to use the same method - guess the form of the solution - as we did in 1st order linear diffneretial equation, then the form of the solution will definitely not $u(x)y + v(x) = C$.

$$(y^m)' = my^{m-1}y',$$

However, let us follow the same idea, which is to guess

3 Laplace Transform

3.1 Definition and a review of partial fraction decomposition

Definition 19 (Laplace Transform)**.** Let $f(t)$ be a function defined for $t \geq 0$. Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

is called the **Laplace Transform** of f.

Example 19.1. Let use calculate the Laplace Transform of function $f(t) = t$,

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t) dt \\
&= \int_0^\infty e^{-st} t dt, \qquad u = st,\ du = sdt,\ u \text{ is considered as a single variable function of } t, \\
&= \int_0^\infty \frac{1}{s^2} e^{-u} u du \\
&= \frac{1}{s^2} \int_0^\infty e^{-u} u du \\
&= \frac{1}{s^2} \left[(-e^{-u} u)\Big|_{u=0}^{u=\infty} + \int_0^\infty e^{-u} du \right] \\
&= \frac{1}{s^2} \left[(0-0) + (-e^{-u})\Big|_{u=0}^{u=\infty} \right] \\
&= \frac{1}{s^2}.
\end{aligned}$$

Remark 12. The example above indicates that the Laplace transform of a function $f(t)$ is a function of another variable s. Basically, the laplace transform transforms a function to another function. So a lot of times we write

$$F(s) = \mathcal{L}\{f(t)\} \qquad \text{for convenience.}$$

More examples,

$$G(s) = \mathcal{L}\{g(t)\}, \qquad Y(s) = \mathcal{L}\{y(t)\}.$$

Example 19.2. Calculate the Laplace transform of $y(t) = t + e^t$,

$$\begin{aligned}
Y(s) = \mathcal{L}\{y(t)\} &= \int_0^\infty e^{-st}(t + e^t)dt \\
&= \int_0^\infty e^{-st} t dt + \int_0^\infty e^{-st} e^t dt \\
&= \frac{1}{s^2} + \int_0^\infty e^{-(s-1)t} dt \\
&= \frac{1}{s^2} + (-\frac{1}{s-1}) e^{-(s-1)t} \Big|_{t=0}^{t=\infty} \\
&= \frac{1}{s^2} + (-\frac{1}{s-1})(0 - 1) \\
&= \frac{1}{s^2} + \frac{1}{s-1}.
\end{aligned}$$

Remark 13. Linear

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s).$$

Remark 14.

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, n = 1, 2, 3, \cdots$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad \mathcal{L}\{\sin(kt)\} = \frac{k}{s^2 + k^2}$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2 + k^2}$$

Remark 15 (Partial Fraction Decomposition)**.** It is covered in Cal 2, please review your old notes, or you can easily find material online.

3.2 Inverse Transform and Transforms of Derivatives

Definition 20 (Inverse Laplace Transform)**.** The inverse Laplace transform is defined and written as

$$f(t) = \mathcal{L}^{-1}\{F(s)\},$$

where $F(s)$ is the Laplace transform of $f(t)$.

Remark 16. Some common inverse Laplace transform

$$\mathcal{L}^{-1}\{\frac{1}{s}\} = 1 \qquad \mathcal{L}^{-1}\{\frac{n!}{s^{n+1}}\} = t^n, n = 1, 2, 3, \cdots$$

$$\mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at} \qquad \mathcal{L}^{-1}\{\frac{k}{s^2 + k^2}\} = \sin(kt)$$

$$\mathcal{L}^{-1}\{\frac{s}{s^2 + k^2}\} = \cos(kt)$$

Definition 21 (Transform of Derivatives)**.** The Laplace transform of the 1st and 2nd order derivatives can be given as the following

$$\begin{aligned} \text{1st order derivative:} \quad & \mathcal{L}\{f'(t)\} = sF(s) - f(0), \\ \text{2nd order derivative:} \quad & \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0). \end{aligned}$$

Note. *For higher order derivatives, there is the general formula*

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

Method 21.1 (Solve Initial Value Problem by using Laplace transform)**.** We can follow the steps:

1. Apply Laplace transform on both side of the D.E., make use of the initial values while finding the Laplace transform of derivatives.

2. Solve for $Y(s) = \mathcal{L}\{y(t)\}$, normally $Y(s)$ will be a fraction, do partial fraction decomposition so that you can find inverse Laplace transform.

3. Apply inverse Laplace transform of $Y(s)$ so that we can get the solution $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

3.3 Operational Properties

Definition 22 (Unit Step Function)**.** The unit step function $\mathcal{U}(t-a)$ is defined as

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a, \\ 1, & t \ge 1. \end{cases}$$

Note: $\mathcal{U}(-) = 0$ and $\mathcal{U}(+\,\text{or}\,0) = 1$.

Method 22.1 (Rewrite a piecewise defined function in terms of unit step functions)**.** If a function is piecewise defined as the following

$$f(t) = \begin{cases} f_1(t), & 0 \le t < a, \\ f_2(t), & a \le t < b, \\ f_3(t), & t \ge b, \end{cases}$$

we start with the assumption

$$f(t) = A\mathcal{U}(t-a) + B\mathcal{U}(t-b) + C,$$

then we can determine A, B and C, by

- $0 \le t < a, \quad f_1(t) = C,$
- $a \le t < b, \quad f_2(t) = A + C,$
- $t \ge b, \quad f_3(t) = A + B + C.$

Definition 23 (Operational Property 1)**.**

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s), \quad F(s) = \mathcal{L}\{f(t)\}.$$

Definition 24 (Operational Property 2)**.**

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad F(s) = \mathcal{L}\{f(t)\}.$$

Definition 25 (Operational Property 3)**.**

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s), \quad F(s) = \mathcal{L}\{f(t)\}.$$

Definition 26 (Operational Property 4)**.**

$$\mathcal{L}\{f * g\} = F(s)G(s), \quad F(s) = \mathcal{L}\{f(t)\},\ G(s) = \mathcal{L}\{g(t)\}.$$

Here $f * g$ is the convolution of functions f and g, its defined by

$$f * g(s) = \int_0^t f(x)g(t-x)dx.$$

Definition 27 (Operational Property 5)**.** If f is a periodic function with period k, i.e., $f(t+k) = f(t)$, then its Laplace transform can be given by

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{sk}} \int_0^k e^{-st}f(t)dt.$$

4 Math Modeling I

4.1 Mixture

If we use $A(t)$ to denote the amount of salt in the tank at time t, then the derivative $\frac{dA(t)}{dt}$ of $A(t)$ means the amount of salt changed in tank per minute. In general: the informations you get will be

- amount of liquid in tank initially: g_0 (gallon)
- amount of salt in tank initially: A_0 (lb or gram)
- concentration of salt in liquid pumping in: c_{in} (lb/gal or gram/gal)
- amount of liquid pumping in per minute: L_{in} (gal/min)
- amount of liquid pumping out per minute: L_{out} (gal/min)
- concentrtion of salt in liquid pumping out, which is the same as concentration of salt in tank: c_{out} (lb/gal or gram/gal)

In general: the differential equation will be given by

$$\frac{dA(t)}{dt} = c_{in} \cdot L_{in} - c_{out} \cdot L_{out}, \quad A(0) = A_0,$$

where c_{out} normally is not given,

$$c_{out} = \frac{A(t)}{g_0 + L_{in} \cdot t - L_{out} \cdot t},$$

you can always use the method of 1st order linear differential equation to solve the differential equation.

4.2 Revisit 'proportional'

- A is 'proportional' to B $\Rightarrow$ $A = kB$,
- A is 'proportional' to B and C $\Rightarrow$ $A = kBC$,
- Type of questions may show: Temerature, Population, Virus spread (logistic growth), Chemical reaction.
- Method of solving D.E. in this type question: they are **separable differential equations**.

4.3 Series Circuit

Kirchorff's second law:

$$Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = E(t),$$

in addtion, $q'(t) = i(t)$. $i(t)$ is the current, $q(t)$ is the charge. For this type of question, you normally will get a second order differential equation, which we can only use Laplace transform to solve it. We will learn other methods to solve higher order differential equations in the coming week.

5 Higher Order Linear Differential Equations

5.1 Preliminary Theory

Definition 28. Order of differential equation, number of initial/boundary condition.

Definition 29 (Higher Order Linear Homogeneous D.E.)**.** A nth order linear differential equation is homogeneous if it can be written in the following form

$$a_n(x)y^{(n)} + \cdot + a_1(x)y' + a_0(x)y = 0.$$

Definition 30 (Higher Order Linear Non-Homogeneous D.E.)**.** A nth order linear differential equation is non-homogeneous if it can be written in the following form

$$a_n(x)y^{(n)} + \cdot + a_1(x)y' + a_0(x)y = f(x), \quad \text{where } f(x) \text{ is not a zero function.}$$

Definition 31 (Function Linear Dependence/Independence)**.** A set of n functions are linearly dependent is there are n constants $c_1, \dots, c_n$ such that at least one of them is not zero, and satisfies equation

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0.$$

n functions are linearly independent if they are not linearly dependent.

Remark 17. It is equivalent to say that the n functions $f_1(x), \dots, f_n(x)$ are linearly dependent if we can express one of the functions as a linear combination of the others. For example, the three function $f_1(x) = 1, f_2(x) = \cos^2 x$ and $f_3(x) = \sin^2 x$ are linearly dependent because $f_1 = f_2 + f_3$.

Definition 32. We can use the Wronskian matrix to determine the linear dependency of functions. The Wronskian matrix of n functions $f_1(x), \dots, f_n(x)$is definded as

$$W(f_1, \dots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

The n functions are linearly dependent is equivalent to $W(f_1, \dots, f_n) = 0$; the n functions are linearly independent is equivalent to $W(f_1, \dots, f_n) \neq 0$.

Remark 18. The $W(f_1, \dots, f_n) \neq 0$ means that the determinant is not a ZERO FUNCTION.

Definition 33 (Superposition Principle for Homogeneous Linear D.E.)**.** If $y_1(x), \dots, y_m(x)$ are solutions of the a linear homogeneous differential equation

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = 0,$$

then any linear combination

$$y(x) = C_1 y_1(x) + \cdots + C_m y_m(x), \qquad C_1, \dots, C_m \text{ are constants,}$$

of the known solutions $y_1, \dots, y_m$ is still a solution of the differential equation.

Definition 34 (Fundamental Set of Solutions and general solution)**.** For a nth order homogeneous linear differential equation

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = 0,$$

its fundamental set of solution consist of n linearly independent functions that are solutions of the D.E. The general solution of the D.E. is given by the linear combination of the functions from the fundamental set of solution, i.e., if $\{y_1(x), \ldots, y_n(x)\}$ is a fundamental set of solution, then the general solution is

$$y(x) = C_1 y_1(x) + \cdots + C_n y_n(x).$$

Definition 35 (Particular Solution)**.** We call any solution of a non-homogeneous linear differential equation a particular solution.

Definition 36 (Superposition Principle for Non-Homogeneous Linear D.E.)**.** If we have the following particular solutions for each differential equation

$$\begin{aligned}
y_{p_1}(x) \text{ is a particular solution of } \quad & a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = f_1(x), \\
y_{p_2}(x) \text{ is a particular solution of } \quad & a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = f_2(x), \\
\cdots\cdots \quad & \cdots \\
y_{p_m}(x) \text{ is a particular solution of } \quad & a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = f_m(x),
\end{aligned}$$

then the summation of the particular solutions

$$y_p(x) = y_{p_1}(x) + \cdots + y_{p_m}(x)$$

is a particular solution of

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = f_1(x) + f_2(x) + \cdots + f_m(x).$$

Remark 19. Some n in the content denotes the order of derivatives, some n or m denote the number of objects, be cautious. Most of the time, we deal with differential equations with order less or equal to 3, and number of functions less or equal to 3.

5.2 Homogeneous Linear D.E. with Constant Coefficients

Definition 37 (2nd order)**.** A second order homo. linear D.E.

$$ay'' + by' + cy = 0,$$

let m_1, m_2 be the two roots of its auxiliary equation

$$am^2 + bm + c = 0.$$

If $m_1 \neq m_2$, then $\{e^{m_1 x}, e^{m_2 x}\}$ is a fundamental set of solution; if $m_1 = m_2$, then $\{e^{m_1 x}, xe^{m_1 x}\}$ is a fundamental set of solution;if m_1, m_2 are complex numbers, wait for the coming week.

5.3 Undetermined Coefficients

Associated homogeneous (linear) D.E. = a.h.D.E.

Method 37.1. For a linear differential equation with constant coefficients

$$ay'' + by' + cy = f(x),$$

1. solve its a.h.D.E. to get the fundamental set of solution and y_c;
2. make initial assumption y_{p_0} of the form of a particular solution;
3. modify the initial assumption by comparing terms with functions in the fundamental set of solution, denote the modified assumption as y_p;
4. put your modified assumption y_p to the original D.E. to solve values for the undetermined coefficients.

6 Linear System of Differential Equations

In the preliminary section, the theory applies to linear system of n differential equations. But we will only focus on homogenous linear system of two differential equations in practice.

6.1 All you need to know about matrix in this class

• The only two matrices you will need:

1. A 2×2 matrix, i.e., a matrix that has two rows and two columns:
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$
2. A 2×1 matrix, i.e., a matrix with two rows and one column:
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$
it is often called a vertical vector.

For matrix more than one column and more than one row, the first index indicates the row location of the element, and the second index indicates the column location.

• Matrix multiplication: we will only need the case of a 2×2 matrix multiply a 2×1 matrix:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 1_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix},$$

the result of the right hand side is a 2×1 matrix.

We can not swith the order of matrix multiplication. In general, the element in the resulting matrix at position mn is given by the dot product of the mth row vector of the first matrix and the nth column vector of the second vector. i.e. ,$(a_{ij})_{m\times r} \cdot (b_{jk})_{r\times n} = (\sum_{j=1}^{r} a_{ij}b_{jk})_{m\times n}$ *in general.*