Week 10 Practice Solution

November 9, 2019

- Find an appropriate assumption of the form of a particular solution for the following non-homogenuous linear differential equations with constant coefficients.

1. $y'' - 4y = (x^2 - 3)\sin 2x$;

   SOLUTION

   1). Find general solution of $y'' - 4y = 0$, $m^2 - 4 = 0$, $m_1 = 2$ and $m_2 = -2$. The fundamental set of solution is $\{e^{2x}, e^{-2x}\}$, general solution $y_c = C_1 e^{2x} + C_2 e^{-2x}$.

   2). Initial assumption $y_{p_0} = A\sin 2x + Bx\sin 2x + Cx^2\sin 2x + D\cos 2x + Ex\cos 2x + Fx^2\cos x$. (An easy way to find an initial assumption of this kind of function $f(x) = (x^2 - 3)\sin 2x = h(x)g(x)$ is to calculate derivatives of $h(x)$ and $g(x)$ separately, and get the linearly independent functions by 'pairing'. More specifically, the linearly independent terms from differentiating $h(x) = x^2 - 3$ are $\{x^2, x, 1\}$, and the linearly independent terms from differentiating $g(x)$ are $\{\sin 2x, \cos 2x\}$ By 'pairing' these two sets of linearly independent functions, we get a set of six linearly independent functions $\{x^2\sin 2x, x\sin 2x, \sin 2x, x^2\cos 2x, x\cos 2x, \cos 2x\}$, the initial assumption is given by the linearly combination of the six functions.)

   3). There is no duplication between y_{p_0} and y_c. (i.e., there is no common element between $\{e^{2x}, e^{-2x}\}$ and $\{x^2\sin 2x, x\sin 2x, \sin 2x, x^2\cos 2x, x\cos 2x, \cos 2x\}$.) There is no need to modify it, so an appropriate assumption is $y_p = A\sin 2x + Bx\sin 2x + Cx^2\sin 2x + D\cos 2x + Ex\cos 2x + Fx^2\cos x$.

2. $y'' - y' = -3$;

   SOLUTION

   1). $y'' - y' = 0$,$m^2 - m = 0$, $m(m-1) = 0$, $m_1 = 0$ and $m_2 = 1$. The fundamental set of solution is $\{1, e^x\}$, $y_c = C_1 + C_2 e^x$.

   2). Initial assumption $y_{p_0} = A$. ($f(x) = -3$, $f'(x) = 0$, only constant term.)

3). Compare $y_{p_0} = A \cdot 1$ with $y_c = C_1 \cdot 1 + C_2 e^x$, the constant term $\{1\}$ is duplicated, so we modify it by multiply x, $xy_{p_0} = Ax$, and the duplication is eliminated. So an appropriate assumption is $y_p = Ax$.

3. $y'' + 2y' + y = \sin x + 3\cos(2x)$;

SOLUTION $y_p = A\sin x + B\cos x + C\sin(2x) + D\cos(2x)$.

4. $y''' - 6y'' = 2 - \cos x$;

SOLUTION

1). $y_c = C_1 + C_2 x + C_3 e^{6x}$, i.e., the fundamental set of solution is $\{1, x, e^{6x}\}$;

2). Let $f_1(x) = 2$, $f_2(x) = -\cos x$, get their appropriate assumption separately.

2.1) assumption for $f_1(x) = 2$ is $y_{p_1'} = A$, duplicate with $1 \in \{1, x, e^{6x}\}$. Try $xy_{p_1'} = Ax$, still duplicate with $x \in \{1, x, e^{6x}\}$; try $x^2 y_{p_1'} = Ax^2$, no more duplication with $\{1, x, e^{6x}\}$. So the appropriate assuption for $y''' - 6y'' = f_1(x) = 2$ is $y_{p_1} = Ax^2$;

2.2) assumption for $f_2(x) = -\cos x$ is $y_{p_2'} = A\sin x + B\cos x$. Is there duplication with y_c ? Luckily no. So the initial assumption does not need to be modified, $y_{p_2} = A\sin x + B\cos x$.

3). Since $y''' - 6y'' = f_1(x) + f_2(x)$, by superposition principle, an appropriate assumption is $y_p = y_{p_1} + y_{p_2} = Ax^2 + B\sin x + C\cos x$.

5. $y''' - 2y'' + y' = (x + x^3)e^x$.

SOLUTION

1). $y_c = C_1 + C_2 e^x + C_3 x e^x$, i.e., the fundamental set of solution of the a.h.D.E. is $\{1, e^x, xe^x\}$.

2). Write the differential equation as $y''' - 2y'' + y' = xe^x + x^3 e^x = f_1(x) + f_2(x)$, where $f_1(x) = xe^x$ and $f_2(x) = x^3 e^x$. Similar to question 4, let us find their appropriate assumption separately.

2.1) assumption for $f_1(x) = xe^x$, $y_{p_1'} = A\underline{e^x} + B\underline{xe^x}$, duplication with y_c? yes. Try $xy_{p_1'} = A\underline{xe^x} + Bx^2 e^x$, duplication with y_c? yes (the underline part of $xy_{p_1'}$). Try $x^2 y_{p_1'} = Ax^2 e^x + Bx^3 e^x$, duplication with y_c? no!!! So $x^2 y_{p_1'}$ is a good modification!!! The appropriate assumption for $y''' - 2y'' + y' = f_1(x)$ is $y_{p_1} = Ax^2 e^x + Bx^3 e^x$.

2.2) $y_{p_2'} = A\underline{e^x} + B\underline{xe^x} + Cx^2 e^x + Dx^3 e^x$, duplication with y_c? yes. Try $xy_{p_2'} = A\underline{xe^x} + Bx^2 e^x + Cx^3 e^x + Dx^4 e^x$, duplication? yes. Try $x^2 y_{p_2'} = Ax^2 e^x + Bx^3 e^x + Cx^4 e^x + Dx^5 e^x$, duplication? no!!! So $y_{p_2} = Ax^2 e^x + Bx^3 e^x + Cx^4 e^x + Dx^5 e^x$.

3). By superposition principle, $y_p = y_{p_1} + y_{p_2} = Ax^2e^x + Bx^3e^x + Cx^4e^x + Dx^5e^x + Ex^2e^x + Fx^3e^x = \underline{A}x^2e^x + \underline{B}x^3e^x + Cx^4e^x + Dx^5e^x$. (We 'merge' the undetermined coefficients A and E to a new A, it does not hurt the purpose of finding a particular solution.)

QUESTION: Can we use this method to find a particular solution of $y'' - 2y' - 3y = \ln x$? How about $y'' - 3y' - 4y = \frac{1}{x^2}$? Why?

ANSWER: No. For these two differential equations, the first one has $f(x) = \ln x$, $f' = \frac{1}{x}$, $f'' = -\frac{1}{x^2}$, ..., the derivatives of $\ln x$ are not generated by a finite number of linearly independent functions. (In more human language, none of the terms in its derivatives iterates.) Same reason for $y'' - 3y' - 4y = \frac{1}{x^2}$. That's why we mostly(only) play with functions like $\sin(ax)$, $\cos(ax)$, e^{ax} and polynomials in this method.