Week 4 Practice - Solution

September 23, 2019

1. Show the following differential equations are exact, then solve the D.E. or I.V.P.:

   (a) $(\tan x - \sin x \sin y)dx + \cos x \cos y dy = 0$;

   (b) $(5y - 2x)y' - 2y = 0$;

   SOLUTION:

   - (a) We need to show it is exact first. For the $(\tan x - \sin x \sin y)dx + \cos x \cos y dy = 0$,

$$\begin{aligned} M(x,y) &= \tan x - \sin x \sin y, \\ M_y(x,y) &= (\tan x - \sin x \sin y)_y \\ &= (\tan x)_y - \sin x (\sin y)_y \\ &= 0 - \sin x \cos y = -\sin x \cos y, \end{aligned}$$

     and

$$\begin{aligned} N(x,y) &= \cos x \cos y, \\ N_x(x,y) &= (\cos x \cos y)_x \\ &= \cos y (\cos x)_x \\ &= -\cos y \sin x \\ &= M_y(x,y), \qquad \text{so it is exact.} \end{aligned}$$

     Then we solve it

$$\begin{aligned} \int M dx = \int (\tan x - \sin x \sin y) dx &= \int \frac{\sin x}{\cos x} dx - \sin y \int \sin x dx \\ &= -\ln|\cos x| - \sin y(-\cos x) \quad (u\text{-sub}, u = \cos x) \\ &= -\ln|\cos x| + \cos x \sin y \end{aligned}$$

$$\int N dy = \int (\cos x \cos y) dy = \cos x \int \cos y dy = \cos x \sin y$$

     So the solution is

$$-\ln|\cos x| + \cos x \sin y = C. \qquad \text{DON'T FORGET to set it equal to constant C.}$$

   - (b) Rewrite it in differential form

$$(5y - 2x)dy - 2y dx = 0 \qquad \Rightarrow \qquad M(x,y) = -2y,\ N(x,y) = 5y - 2x,$$

$$M_y = (-2y)_y = -2, \quad N_x = (5y - 2x)_x = 0 - 2 = -2,$$

   it is exact since $M_y = N_x$.

$$\int M dx = \int -2y dx = -2xy, \qquad \int N dy = \int (5y - 2x) dy = \frac{5}{2}y^2 - 2xy,$$

   merge terms and set it to the arbitrary constant C:

$$\frac{5}{2}y^2 - 2xy = C.$$

2. Find the general solution of the following D.E.s. Each of them is either exact or can be made exact.

   (a) $\left(\frac{3y^2-t^2}{y^5}\right)\frac{dy}{dt}+\frac{t}{2y^4}=0$;

   (b) $y(x+y+1)dx+(x+2y)dy=0$;

   (c) $(y^2+xy^3)dx+(5y^2-xy+y^3\sin y)dy=0$.

SOLUTION:

• (a) Your inituition may tell you to simplify by multiplying y^5, however, let us check if the D.E. is exact or not before following your instinct. The D.E. in differential form

$$\left(\frac{3y^2-t^2}{y^5}\right)dy+\frac{t}{2y^4}dt=0 \quad\Rightarrow\quad \frac{t}{2y^4}dt+\left(\frac{3y^2-t^2}{y^5}\right)dy=0 \quad\Rightarrow\quad M(t,y)=\frac{t}{2y^4},\ N(t,y)=\left(\frac{3y^2-t^2}{y^5}\right),$$

$$M_y=\left(\frac{t}{2y^4}\right)_y=\frac{t}{2}\left(\frac{1}{y^4}\right)_y=\frac{t}{2}(y^{-4})_y=\frac{-4t}{2y^5}=-\frac{2t}{y^5},$$

$$N_t=\left(\frac{3y^2-t^2}{y^5}\right)_t=\frac{(-t^2)_t}{y^5}=-\frac{2t}{y^5},$$

it is exact since $M_y=N_t$. Now let us solve it

$$\int Mdt=\int\frac{t}{2y^4}dt=\frac{t^2}{4y^4},$$

$$\int Ndy=\int\frac{3y^2-t^2}{y^5}dy=\int 3y^{-3}dy-\int t^2y^{-5}dy=\frac{3}{-2}y^{-2}-t^2(-\frac{1}{4})y^{-4}=-\frac{3}{2}y^{-2}+\frac{t^2}{4y^4},$$

merge and set it to C,

$$-\frac{3}{2}y^{-2}+\frac{t^2}{4y^4}=C.$$

• (b) Let us start with calculating the partial derivatives for $M=y(x+y+1)=xy+y^2+y$, $N=x+2y$,

$$M_y=x+2y+1,\qquad N_x=1,$$

check if $\frac{N_x-M_y}{N}$ is in terms of x,

$$\frac{N_x-M_y}{N}=\frac{1-(x+2y+1)}{x+2y}=-1(=-x^0),\qquad \text{this can be considered as a function in terms of } x \text{ only},$$

the integrating factor

$$\mu(x)=e^{-\int(-1)dx}=e^x,$$

the D.E. becomes exact when you integrate $\mu(x)$ on both sides

$$e^x(xy+y^2+y)dx+e^x(x+2y)dy=0,$$

the general solution is

$$\left.\begin{array}{l}\displaystyle\int e^xxy+e^xy^2+e^xydx=y(xe^x-e^x)+y^2e^x+ye^x=yxe^x+y^2e^x\\ \displaystyle\int e^xx+2ye^xdy=e^xxy+y^2e^x\end{array}\right.\quad\Rightarrow\quad e^xxy+e^xy^2=C.$$

• (c) $(y^2+xy^3)dx+(5y^2-xy+y^3\sin y)dy=0$. Determine $M=y^2+xy^3$, $N=5y^2-xy+y^3\sin y$,

$$M_y=(y^2+xy^3)_y=2y+3xy^2\quad\text{and}\quad N_x=(5y^2-xy+y^3\sin y)_x=-y,$$

check if $\frac{N_x-M_y}{N}$ is in terms of x only,

$$\frac{N_x-M_y}{N}=\frac{-y-(2y+3xy^2)}{5y^2-xy+y^3\sin y},\qquad\text{not in terms of } x \text{ only},$$

so let us check if the other fraction $\frac{M_y - N_x}{M}$ is in terms of $y$only,

$$\frac{M_y - N_x}{M} = \frac{2y + 3xy^2 - (-y)}{y^2 + xy^3} = \frac{3y(1+xy)}{y^2(1+xy)} = \frac{3}{y}, \qquad \text{it is in terms of } y \text{ only,}$$

intergrating factor

$$\mu(y) = e^{-\int \frac{3}{y} dy} = e^{-3\ln|y|} = y^{-3},$$

multiply it to make the equation exact

$$(y^{-1} + x)dx + (5y^{-1} - xy^{-2} + \sin y)dy = 0,$$

the solution is

$$\left.\begin{aligned} \int (y^{-1} + x)dx &= xy^{-1} + \frac{1}{2}x^2 \\ \int (5y^{-1} - xy^{-2} + \sin y)dy &= 5\ln|y| + xy^{-1} - \cos y \end{aligned}\right\} \Rightarrow \quad xy^{-1} + 5\ln|y| + \frac{1}{2}x^2 - \cos y = C.$$

3. Solve the following I.V.P.

$$(x+y)^2 dx + (2xy + x^2 - 1)dy = 0, \qquad y(1) = 1.$$

SOLUTION: It is a first order differential equation, however it is clearly not linear and separable, so we wish it is exact. To check if it is exact or not, let us calculate the derivatives first,

$$M_y = (x^2 + 2xy + y^2)_y = 2x + 2y. \qquad N_x = (2xy + x^2 - 1)_x = 2y + 2x,$$

they are the same which means it is an exact differential equation.
Solve it to get its general solution first

$$\left.\begin{aligned} \int (x^2 + 2xy + y^2)dx &= \frac{1}{3}x^3 + x^2y + xy^2 \\ \int (2xy + x^2 - 1)dy &= xy^2 + x^2y - y \end{aligned}\right\} \Rightarrow \quad \frac{1}{3}x^3 + x^2y + xy^2 - y = C,$$

using initial condition to get value of C

$$C = \frac{1}{3} + 1 + 1 - 1 = \frac{4}{3}.$$

The solution to the I.V.P. is

$$\frac{1}{3}x^3 + x^2y + xy^2 - y = \frac{4}{3}.$$

4. Solve the following D.E. by using two different methods,

$$x\frac{dy}{dx} = 2e^x - y + 6x^2.$$

SOLUTION:

• 1st method: using the formula for 1st order linear D.E.

• 2nd method: this D.E. is exact because it is equivalent to

$$(y - 2e^x - 6x^2)dx + xdy = 0,$$

so we have

$$M_y = (y - 2e^x - 6x^2)_y = 1 \qquad N_x = (x)_x = 1,$$

so the solution is

$$\left.\begin{aligned} \int (y - 2e^x - 6x^2)dx &= xy - 2e^x - 2x^3 \\ \int xdy &= xy \end{aligned}\right\} \Rightarrow \quad xy - 2e^x - 2x^3 = C.$$

5. Solve the following differential equations by using an appropriate substitution (homogeneous, Bernoulli's or reductible):

   (a) $x\frac{dy}{dx} + y = \frac{1}{y^2}$;

   (b) $\frac{dy}{dx} - y = e^x y^2$;

   (c) $\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$;

      - SOLUTION: let $u = y-2x+3$, $du = (y-2x+3)_x dx + (y-2x+3)_y dy = -2dx + dy$, so $dy = du + 2dx$,

$$\frac{du + 2dx}{dx} = 2 + \sqrt{u} \quad \Rightarrow \quad \frac{du}{dx} = u^{\frac{1}{2}} \quad \Rightarrow \quad u^{-\frac{1}{2}} du = dx$$

      integrate to get general solution

$$2u^{\frac{1}{2}} = x + C, \quad 2\sqrt{y - 2x + 3} = x + C.$$

   (d) $\frac{dy}{dx} = \frac{x+3y}{3x+y}$;

   (e) $-ydx + (x + \sqrt{xy})dy = 0$;

   (f) $\frac{dy}{dx} = \frac{1-x-y}{x+y}$.

      - SOLUTION: let $u = x + y$, $du = dx + dy$, so $dy = du - dx$,

$$\frac{du - dx}{dx} = \frac{1-u}{u} \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{u} \quad \Rightarrow \quad udu = dx$$

      so the general solution is

$$\frac{1}{2}u^2 = x + C, \qquad \frac{1}{2}(x+y)^2 = x + C.$$

- THANK EVERYONE FOR HELPING TO CREAT A PERFECT SOLUTION!!!

1. Challenge Question (NOT on Quiz nor Exam) (Recommended for honor program)

   (a) Are separable D.E.s exact?

      - SOUTION: YES. Because $\frac{dy}{dx} = g(x)h(y)$ is equivalent to

$$\frac{1}{h(y)}dy = g(x)dx \qquad \Leftrightarrow \qquad g(x)dx + (-\frac{1}{h(y)})dy = 0,$$

      where $M = g(x)$ and $N = -\frac{1}{h(y)}$,

$$M_y = \frac{\partial}{\partial y}g(x) = 0, \quad N_x = \frac{\partial}{\partial x}\frac{1}{h(y)} = 0.$$

   (b) Can we use the method of making non-exact D.E.s exact to dedrive the formula for homogeneous, Bernoulli's and reductible D.E.s? Why or Why not? (Don't try too hard and spend too much time).

   (c) Dedrive the formula for homogeneous, and think about why being homogeneous is an important property (you can google it).

   (d) Try to dedrive the formula of making non-exact exact on your own (trying it on your own will make a huge difference than just understanding your notes).