Week 5 Practice (Homogenous and Euler's Method)

September 23, 2019

1. Use Euler's Method to approximate the value of the indicated solution, show your steps.
   (a) $y' = (x - y)^2, y(0) = 0.5, \quad y(1) = ?, \quad n = 2;$
   (b) $y' = y - y^2, y(1) = 2, \quad y(1.6) = ?, \quad h = 0.2.$

   SOLUTION:

   - (a) The initial value $y(0) = 0.5$ tells us that $x_0 = 0$ and $y_0 = 0.5$. The value, $y(1)$, we are looking for gives $a = 1$. Instead of given h, this question tells us how many steps we want to use to get the approximation, which is two steps, from the relation $n = \frac{a-x_0}{h}$, we get the step size
   $$h = \frac{a - x_0}{n} = \frac{1-0}{2} = 0.5.$$
   And y_2 is the value we want. Start the recursing process:
     - $m = 0$: $y_1 = y_0 + hf(x_0, y_0) = y_0 + h \times (x_0 - y_0)^2 = 0.5 + 0.5 \times (0 - 0.5)^2 = 0.625$, $x_1 = x_0 + h = 0.5$;
     - $m = 1$: $y_2 = y_1 + hf(x_1, y_1) = y_1 + h \times (x_1 - y_1)^2 = 0.625 + 0.5 \times (0.5 - 0.625)^2 = 0.6328$.

     So $y(1) \approx y_2 = 0.6328$.
   - (b) The initial value
   $$y(1) = 2 \quad \Rightarrow \quad x_0 = 1, y_0 = 2.$$
   The D.E.
   $$y' = y - y^2 \quad \Rightarrow \quad f(x, y) = y - y^2.$$
   The desired value $y(1.6)$ and the step size $h = 0.2$ implies
   $$a = 1.6, \quad n = \frac{a - x_0}{h} = \frac{1.6 - 1}{0.2} = 3, \quad y_3 \text{ is the value we are looking for.}$$
   Start the recrusing process:
     - $m = 0$: $y_1 = 2 + 0.2 \times (2 - 2^2) = 1.6$, $x_1 = x_0 + h = 1.2$;
     - $m = 1$: $y_2 = 1.6 + 0.2 \times (1.6 - 1.6^2) = 1.408$, $x_2 = x_1 + h = 1.4$;
     - $m = 2$: $y_3 = 1.408 + 0.2 \times (1.408 - 1.408^2) = 1.2931$.

     So $y(1.6) \approx y_3 = 1.2931$.

2. Verify the following differential equations are homogeneous and solve them.
   (a) $\frac{dy}{dx} = \frac{x+3y}{3x+y}$;
   (b) $-ydx + (x + \sqrt{xy})dy = 0$;

   SOLUTION:

   - (a) $M(x, y) = x + 3y$, $N(x, y) = -(3x + y)$.
   $$M(tx, ty) = tx + 3ty = t(x + 3y) = t^1 M(x, y) \qquad \text{it is homogeneous of degree 1.}$$

Similarly, $N(tx, ty) = tN(x, y)$ is homogeneous of degree 1. The differential equation is homogeneous. Do substitution $y = ux$, $dy = udx + xdu$,

$$\frac{udx + xdu}{dx} = \frac{x + 3ux}{3x + ux} \quad \Rightarrow \quad u + x\frac{du}{dx} = \frac{1 + 3u}{3 + u} \quad \Rightarrow \quad x\frac{du}{dx} = \frac{1 - u^2}{3 + u},$$

$$\Rightarrow \quad \frac{3 + u}{1 - u^2}du = \frac{1}{x}dx \quad \Rightarrow \quad (\frac{1}{1 + u} + \frac{2}{1 - u})du = \frac{1}{x}dx \quad \Rightarrow \quad \ln|1 + u| - 2\ln|u - 1| = \ln|x| + C,$$

where the above step used the fact $1 - u^2 = (1 - u)(1 + u)$ and partial fraction decomposition (not on exam 1).

- (b) See lecture notes Example 18.2 and Example 18.3.