Week 6 - Practice

October 6, 2019

1. Find the Laplace transform $\mathcal{L}\{f(t)\}$ of the following functions by definition,

   (a) $f(t) = \begin{cases} -1, & 0 \le t < 1, \\ 1, & t \ge 1. \end{cases}$

   (b) $f(t) = \begin{cases} 2t+1, & 0 \le t < 1, \\ 0, & t \ge 1. \end{cases}$

   (c) $f(t) = e^{t+7}$,

   (d) $f(t) = \cos t$.

   SOLUTION:

   - $f(t) = \begin{cases} -1, & 0 \le t < 1, \\ 1, & t \ge 1. \end{cases}$

$$\begin{aligned}
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt &= \int_0^1 e^{-st} \cdot (-1)dt + \int_1^\infty e^{-st} \cdot 1dt \\
&= -1 \cdot \left(-\frac{1}{s}\right) e^{-st}\Big|_{t=0}^{t=1} + \left(-\frac{1}{s}\right) e^{-st}\Big|_{t=1}^{t=\infty} \\
&= \frac{1}{s}\left(e^{-s} - e^{0}\right) - \frac{1}{s}\left(e^{-\infty} - e^{-s}\right) \\
&= \frac{1}{s}e^{-s} - \frac{1}{s} + \frac{1}{s}e^{-s} = \frac{2}{s}e^{-s} - \frac{1}{s}.
\end{aligned}$$

   - $f(t) = \begin{cases} 2t+1, & 0 \le t < 1, \\ 0, & t \ge 1. \end{cases}$

$$\begin{aligned}
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt &= \int_0^1 e^{-st} \cdot (2t+1)dt + \int_1^\infty e^{-st} \cdot 0dt \\
&= 2\int_0^1 e^{-st} \cdot tdt + \int_0^1 e^{-st}dt \\
&= 2\left[\left(-\frac{1}{s}\right) e^{-st}t\Big|_{t=0}^{t=1} + \frac{1}{s}\int_0^1 e^{-st}dt\right] + \left(-\frac{1}{s}\right) e^{-st}\Big|_{t=0}^{t=1} \\
&= 2\left[\left(-\frac{1}{s}\right)\left(e^{-s} - 0\right) + \left(-\frac{1}{s^2}\right) e^{-st}\Big|_{t=0}^{t=1}\right] + \left(-\frac{1}{s}\right)\left(e^{-s} - 1\right) \\
&= 2\left[-\frac{1}{s}e^{-s} - \frac{1}{s^2}(e^{-s} - 1)\right] - \frac{1}{s}e^{-s} + \frac{1}{s} \\
&= -\frac{2}{s^2}e^{-s} - \frac{3}{s}e^{-s} + \frac{2}{s^2} + \frac{1}{s}
\end{aligned}$$

   note: 2nd-3rd line used integration by parts.

- $f(t) = e^{t+7}$

$$\begin{aligned}
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt &= \int_0^\infty e^{-st} \cdot e^{t+7} dt \\
&= \int_0^\infty e^{-st+t} \cdot e^7 dt \qquad \text{(note: } e^7 \text{ is a constant)} \\
&= e^7 \int_0^\infty e^{-(s-1)t} dt \\
&= e^7 \cdot \left(-\frac{1}{s-1}\right) e^{-(s-1)t}\Big|_{t=0}^{t=\infty} \\
&= e^7 \cdot \left(-\frac{1}{s-1}\right) (e^{-\infty} - e^0) = \frac{e^7}{s-1}.
\end{aligned}$$

- $f(t) = \cos t$

$$\begin{aligned}
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt &= \int_0^\infty e^{-st} \cdot \cos t dt \\
&= (e^{-st} \sin t)\Big|_{t=0}^{t=\infty} - \int_0^\infty (-s) e^{-st} \sin t dt \\
&= (0 - e^0 \sin 0) + s \int_0^\infty e^{-st} \sin t dt \\
&= s \left[(-e^{-st} \cos t)\Big|_{t=0}^{t=\infty} - s \int_0^\infty e^{-st} \cos t dt \right] \\
&= s \left[(0 + e^0 \cos 0) - s \int_0^\infty e^{-st} \cos t dt \right] \\
&= s - s^2 \int_0^\infty e^{-st} \cos t dt
\end{aligned}$$

look at the first line and the last line, it means

$$\int_0^\infty e^{-st} \cdot \cos t dt = s - s^2 \int_0^\infty e^{-st} \cdot \cos t dt \qquad \Rightarrow \qquad (1+s^2) \int_0^\infty e^{-st} \cdot \cos t dt = s$$

so

$$\int_0^\infty e^{-st} \cdot \cos t dt = \frac{s}{1+s^2}.$$

2. Find the Laplace transform $\mathcal{L}\{f(t)\}$ of the following functions by the following chart

$1)\ \mathcal{L}\{1\} = \frac{1}{s}$, $\quad 2)\ \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $\quad 3)\ \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, where $n! = 1 \cdot 2 \cdot \dots \cdot n$, $n = 1, 2, 3, \cdots$
$4)\ \mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$, $\quad 5)\ \mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$,

(a) $f(t) = 7t + 3$;
(b) $f(t) = t^2 - e^{-9t} + 5$;
(c) $f(t) = 4t^2 - 5 \sin 3t$;
(d) $f(t) = \cos 5t + \sin 2t$.

SOLUTION:

- $\mathcal{L}\{7t+3\} = 7\mathcal{L}\{t\} + 3\mathcal{L}\{1\} = \frac{7}{s^2} + \frac{3}{s}$
- $\mathcal{L}\left\{t^2 - e^{-9t} + 5\right\} = \mathcal{L}\left\{t^2\right\} - \mathcal{L}\left\{e^{(-9)t}\right\} + 5\mathcal{L}\{1\} = \frac{2}{s^3} - \frac{1}{s-(-9)} + \frac{5}{s} = \frac{2}{s^3} - \frac{1}{s+9} + \frac{5}{s}$
- $\mathcal{L}\left\{4t^2 - 5\sin(3t)\right\} = 4\mathcal{L}\left\{t^2\right\} - 5\mathcal{L}\{\sin(3t)\} = \frac{4 \cdot 2!}{s^{2+1}} - \frac{5 \cdot 3}{s^2+3^2} = \frac{8}{s^3} - \frac{15}{s^2+9}$
- $\mathcal{L}\{\cos(5t) + \sin(2t)\} = \mathcal{L}\{\cos(5t)\} + \mathcal{L}\{\sin(2t)\} = \frac{s}{s^2+5^2} + \frac{2}{s^2+2^2} = \frac{s}{s^2+25} + \frac{2}{s^2+4}$

3. Find the following inverse Laplace transform.

(a) $\mathcal{L}^{-1}\{\frac{1}{s^3}\}$;

(b) $\mathcal{L}^{-1}\{\frac{s+1}{s^2-4s}\}$;

(c) $\mathcal{L}^{-1}\{\frac{2s-4}{(s^2+s)(s^2+1)}\}$;

(d) $\mathcal{L}^{-1}\{\frac{10s}{s^2+16}\}$.

SOLUTION:

- $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \frac{t^2}{2!} = \frac{1}{2}t^2$
- $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{s(s-4)}\right\}$. Apply partial fraction decomposition to

$$\frac{s+1}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4} = \frac{A(s-4)+Bs}{s(s-4)} \qquad \Rightarrow \qquad s+1 = A(s-4)+Bs,$$

take special values for s to find A and B,

$$s=4, \quad 4+1 = A(4-4)+B\cdot 4, \qquad \Rightarrow \qquad B = \frac{5}{4},$$
$$s=0, \quad 0+1 = A(0-4)+B\cdot 0, \qquad \Rightarrow \qquad A = -\frac{1}{4},$$

so the inverse transform

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s(s-4)}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{4}\frac{1}{s} + \frac{5}{4}\frac{1}{s-4}\right\} = -\frac{1}{4}\cdot 1 + \frac{5}{4}e^{4t} = -\frac{1}{4} + \frac{5}{4}e^{4t}$$

- $\mathcal{L}^{-1}\{\frac{2s-4}{(s^2+s)(s^2+1)}\} = \mathcal{L}^{-1}\left\{\frac{2s-4}{s(s+1)(s^2+1)}\right\}$. Partial fraction decomposition

$$\frac{2s-4}{s(s+1)(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1} = \frac{A(s+1)(s^2+1)+Bs(s^2+1)+(Cs+D)s(s+1)}{s(s+1)(s^2+1)}$$

so

$$2s-4 = A(s+1)(s^2+1)+Bs(s^2+1)+(Cs+D)s(s+1)$$

take special values for $s$to find A and B,

$$\begin{aligned} s=-1, &\quad 2(-1)-4 = 0+B(-1)[(-1)^2+1]+0 &\Rightarrow&\quad B = 3,\\ s=0, &\quad 2\cdot 0-4 = A+0+0 &\Rightarrow&\quad A = -4,\\ s=1, &\quad -2 = \mathbf{4}A+2B+2(C+D) &\Rightarrow&\quad C+D = \mathbf{4},\\ s=2, &\quad 2\cdot 2-4 = 15A+10B+6(2C+D) &\Rightarrow&\quad 2C+D = 5, \end{aligned}$$

the last two equations gives values of C and D:

$$C = (2C+D)-(C+D) = 5-\mathbf{4} = \mathbf{1}, \qquad D = \mathbf{4}-\mathbf{1} = \mathbf{3}.$$

So the inverse Laplace transform is

$$\mathcal{L}^{-1}\left\{\frac{2s-4}{s(s+1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{-\frac{4}{s} + \frac{3}{s+1} + \frac{\mathbf{1}s}{s^2+1} + \frac{\mathbf{3}}{s^2+1}\right\} = -4+3e^{-t}+\cos t+\mathbf{3}\sin t.$$

- $\mathcal{L}^{-1}\{\frac{10s}{s^2+16}\} = 10\mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} = 10\cos(4t)$.

4. Use Laplace transform to solve the following I.V.P.

(a) $y' - y = 2\cos 5t, \quad y(0) = 0$;

(b) $y' + 6y = e^{4t}, \quad y(0) = 2$;

(c) $y' - y = 1, \quad y(0) = 0$.

SOLUTION:

- $y' - y = 2\cos 5t, \quad y(0) = 0;$

$$\begin{aligned}
\mathcal{L}\{y' - y\} &= \mathcal{L}\{2\cos(5t)\} \\
\mathcal{L}\{y'\} - \mathcal{L}\{y\} &= 2\mathcal{L}\{\cos(5t)\} \\
sY(s) - y(0) - Y(s) &= \frac{2\cdot s}{s^2+5^2} \\
(s-1)Y(s) - 0 &= \frac{2s}{s^2+25} \\
Y(s) &= \frac{2s}{(s-1)(s^2+25)}
\end{aligned}$$

to get the inverse Laplace transform, let us decompose

$$\frac{2s}{(s-1)(s^2+25)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+25} = \frac{A(s^2+25)+(Bs+C)(s-1)}{(s-1)(s^2+25)}$$

$$2s = A(s^2+25) + (Bs+C)(s-1)$$

$$\begin{aligned}
&s=1, && 2 = A(1+25)+0, && \Rightarrow && A = \frac{1}{13}, \\
&s=0, && 0 = \frac{1}{13}(0+25) + C\cdot(-1), && \Rightarrow && C = \frac{25}{13}, \\
&s=-1, && -2 = \frac{1}{13}[(-1)^2+25] + (-B+\frac{25}{13})\cdot(-2), && \Rightarrow && B = \frac{25}{13} \mathbf{- 2} = \mathbf{-\frac{1}{13}},
\end{aligned}$$

so the solution to the I.V.P.

$$\begin{aligned}
y(y) = \mathcal{L}^{-1}\left\{\frac{2s}{(s-1)(s^2+25)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{13}\frac{1}{s-1} + \frac{51}{13}\frac{s}{s^2+25} + \frac{25}{13}\frac{1}{s^2+25}\right\} \\
&= \frac{1}{13}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \mathbf{-\frac{1}{13}}\mathcal{L}^{-1}\left\{\frac{s}{s^2+5^2}\right\} + \frac{25}{13}\mathcal{L}^{-1}\left\{\frac{1}{s^2+5^2}\right\} \\
&= \frac{1}{13}e^t \mathbf{-\frac{1}{13}}\cos(5t) + \frac{25}{13}\frac{\sin(5t)}{5} = \frac{1}{13}e^t \mathbf{-\frac{1}{13}}\cos(5t) + \frac{5}{13}\sin(5t)
\end{aligned}$$

- $y' + 6y = e^{4t}, \qquad y(0) = 2;$

$$\begin{aligned}
\mathcal{L}\{y' + 6y\} &= \mathcal{L}\left\{e^{4t}\right\} \\
\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} &= \frac{1}{s-4} \\
sY(s) - y(0) + 6Y(s) &= \frac{1}{s-4} \\
(s+6)Y(s) - 2 &= \frac{1}{s-4} \\
Y(s) &= \frac{1+2(s-4)}{(s-4)(s+6)}
\end{aligned}$$

partial decomposition

$$\frac{1+2(s-4)}{(s-4)(s+6)} = \frac{A}{s-4} + \frac{B}{s+6} = \frac{A(s+6)+B(s-4)}{(s-4)(s+6)}$$

$$\begin{aligned}
&s=4, && 1+0 = A(4+6)+0, && \Rightarrow && A = \frac{1}{10}, \\
&s=-6, && 1+2\cdot(-6-4) = 0 + B\cdot(-6-4), && \Rightarrow && B = \frac{19}{10},
\end{aligned}$$

so the solution is

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1+2(s-4)}{(s-4)(s+6)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{10}\frac{1}{s-4} + \frac{19}{10}\frac{1}{s+6}\right\} = \frac{1}{10}e^{4t} + \frac{19}{10}e^{-6t}.$$

- $y' - y = 1, \qquad y(0) = 0.$

$$\begin{aligned}
\mathcal{L}\{y' - y\} &= \mathcal{L}\{1\} \\
\mathcal{L}\{y'\} - \mathcal{L}\{y\} &= \frac{1}{s} \\
sY(s) - y(0) - Y(s) &= \frac{1}{s} \\
(s-1)Y(s) &= \frac{1}{s} \\
Y(s) &= \frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s} \\
y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s}\right\} = e^t - 1
\end{aligned}$$

5. Chanllenge: the Laplace transform $\mathcal{L}\{f(t)\}$ of the following functions by definition.

   (a) $f(t) = t\cos t$;

   (b) $f(t) = t\sin t$.

   SOLUTION:

   - $f(t) = t\cos t$,

$$\begin{aligned}
\mathcal{L}\{t\cos t\} &= \int_0^\infty e^{-st} t\cos t dt \\
&= (e^{-st} t\sin t)\Big|_{t=0}^{t=\infty} - \int_0^\infty (-se^{-st}t + e^{-st})\sin t dt \\
&= (0-0) + s\int_0^\infty e^{-st} t\sin t dt - \int_0^\infty e^{-st}\sin t dt \\
&= s\int_0^\infty e^{-st} t\sin t dt - \int_0^\infty e^{-st}\sin t dt \\
&= s\int_0^\infty e^{-st} t\sin t dt - \mathcal{L}\{\sin t\} \\
\bullet \quad &= s\int_0^\infty e^{-st} t\sin t dt - \frac{1}{s^2+1} \\
&= s\left[(-e^{-st} t\cos t)\Big|_{t=0}^{t=\infty} - \int_0^\infty (-\cos t)(-se^{-st}t + e^{-st})dt\right] - \frac{1}{s^2+1} \\
&= s\left[-s\int_0^\infty e^{-st} t\cos t dt + \int_0^\infty e^{-st}\cos t dt\right] - \frac{1}{s^2+1} \\
&= -s^2\mathcal{L}\{t\cos t\} + s\mathcal{L}\{\cos t\} - \frac{1}{s^2+1} \\
&= -s^2\mathcal{L}\{t\cos t\} + \frac{s^2}{s^2+1} - \frac{1}{s^2+1}
\end{aligned}$$

   which implies

$$\mathcal{L}\{t\cos t\} = -s^2\mathcal{L}\{t\cos t\} + \frac{s^2-1}{s^2+1}, \qquad \Rightarrow \qquad (s^2+1)\mathcal{L}\{t\cos t\} = \frac{s^2-1}{s^2+1}$$

   so

$$\mathcal{L}\{t\cos t\} = \frac{s^2-1}{(s^2+1)^2}.$$

   - $f(t) = t\sin t$. Look at the bulleted step above,

$$\mathcal{L}\{t\cos t\} = s\mathcal{L}\{t\sin t\} - \frac{1}{s^2+1}$$

   so

$$\mathcal{L}\{t\sin t\} = \frac{1}{s}\left(\mathcal{L}\{t\cos t\} + \frac{1}{s^2+1}\right) = \frac{1}{s}\left(\frac{s^2-1}{(s^2+1)^2} + \frac{1}{s^2+1}\right) = \frac{1}{s}\frac{s^2-1+s^2+1}{(s^2+1)^2} = \frac{2s}{(s^2+1)^2}.$$