Week 7 Practice

October 13, 2019

Formula Chart

- $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$; $\mathcal{L}\{\mathcal{U}(t-a)\} = e^{-as}\frac{1}{s}$.
- $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.
- $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$.
- 1) $\mathcal{L}\{1\} = \frac{1}{s}$, 2) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, 3) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, where $n! = 1 \cdot 2 \cdot \cdots \cdot n$, $n = 1, 2, 3, \cdots$
  4) $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$, 5) $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$,

1. Rewrite the following functions in terms of the unit step functions, and then use the operational property to find its Laplace transform.

   (a) $f(t) = \begin{cases} t, & 0 \le t < 2, \\ 0, & t \ge 2. \end{cases}$

   (b) $f(t) = \begin{cases} 3, & 0 \le t < 1, \\ 1, & 1 \le t < 6, \\ 6, & t \ge 6. \end{cases}$

   SOLUTION:

   - $f(t) = \begin{cases} t, & 0 \le t < 2, \\ 0, & t \ge 2. \end{cases}$ This function is not definied contiuous through out its domain, it breaks when $t = 2$. So we will use the operational property regarding unit step function to get its Laplace transform. Let us rewrite the function in terms of the unit step function first. It breaks when $t = 2$, so the initial assumption will be

$$f(t) = A\mathcal{U}(t-2) + B,$$

determing values for A and B,

$$\begin{aligned} \text{when } 0 \le t < 2, \quad f(t) = t = A\mathcal{U}(-) + B, \qquad t &= A \cdot 0 + B = B, \quad \Rightarrow \quad B = t, \\ \text{when } t \ge 2, \quad f(t) = 0 = A\mathcal{U}(+ \text{ or } 0) + B, \quad 0 &= A \cdot 1 + B, \quad \Rightarrow \quad A = -B = -t. \end{aligned}$$

Therefore,

$$\begin{aligned} f(t) &= -t\mathcal{U}(t-2) + t, \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{-t\mathcal{U}(t-2) + t\} \\ &= \mathcal{L}\{-[(t-2)+2]\mathcal{U}(t-2) + t\} \\ &= \mathcal{L}\{-[(t-2)+2]\mathcal{U}(t-2)\} + \mathcal{L}\{t\} \\ &= \mathcal{L}\{-[(t-2)+2]\mathcal{U}(t-2)\} + \frac{1}{s^2}, \end{aligned}$$

we want to use $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$ for $\mathcal{L}\{-[(t-2)+2]\mathcal{U}(t-2)\}$, then $a = 2$, and $f(t-2) = -[(t-2)+2] = -(t-2) - 2$, so $f(t) = -t - 2$,

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{-t-2\} = -\mathcal{L}\{t\} - 2\mathcal{L}\{1\} = -\frac{1}{s^2} - \frac{2}{s},$$

so

$$\mathcal{L}\{-[(t-2)+2]\mathcal{U}(t-2)\} = \mathcal{L}\{f(t-2)\mathcal{U}(t-2)\} = e^{-2t}(-\frac{1}{s^2} - \frac{2}{s}) + \frac{1}{s^2}.$$

- $f(t)=\begin{cases}3, & 0\le t<1,\\ 1, & 1\le t<6,\\ 6, & t\ge 6.\end{cases}$ The function 'breaks' at $t=1$ and $t=6$, so when rewrite the function in terms of unit step functions, we have assumption

$$f(t)=A\mathcal{U}(t-1)+B\mathcal{U}(t-6)+C.$$

Let us determine the values for A, B and C,

$$\begin{aligned}
\text{when } 0\le t<1,\quad & f(t)=3=A\mathcal{U}(-)+B\mathcal{U}(-)+C=0+0+C, & C=3,\\
\text{when } 1\le t<6,\quad & f(t)=1=A\mathcal{U}(+\text{ or }0)+B\mathcal{U}(-)+C=0+B+C, & A=1-C=1-3=-2,\\
\text{when } t\ge 6,\quad & f(t)=6=A\mathcal{U}(+\text{ or }0)+B\mathcal{U}(+\text{ or }0)+C=A+B+C, & B=6-A-C=6-(-2)-3=5,
\end{aligned}$$

so the function

$$f(t)=-2\mathcal{U}(t-1)+5\mathcal{U}(t-6)+3,$$

the Laplace transform is

$$\mathcal{L}\{f(t)\}=-2\mathcal{L}\{\mathcal{U}(t-1)\}+5\mathcal{L}\{\mathcal{U}(t-6)\}+3\mathcal{L}\{1\}=-e^{-s}\frac{2}{s}+e^{-6s}\frac{5}{s}+\frac{3}{s}.$$

2. Find the inverse Laplace transform of the following functions.

   (a) $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\}$,

   (b) $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$,

   (c) $\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+4}\right\}$.

SOLUTION: all the three functions that we want to do inverse laplace transform to are of the form $e^{-as}F(s)$, so we will use the inverse transform for unit step functions

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\}=e^{-as}F(s),\qquad\Rightarrow\qquad \mathcal{L}^{-1}\{e^{-as}F(s)\}=f(t-a)\mathcal{U}(t-a).$$

- $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\}$.

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\}=\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^3}\right\},\qquad e^{-2s}=e^{-as}\text{ and }\frac{1}{s^3}=F(s)\text{ in the formula,}$$

$$\Rightarrow\qquad a=2,\quad\text{and}\quad f(t)=\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}=\frac{t^2}{2!}=\frac{1}{2}t^2,$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\}=\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^3}\right\}=f(t-2)\mathcal{U}(t-2)=\frac{1}{2}(t-2)^2\mathcal{U}(t-2).$$

- $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$.

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}=\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{s^2+1}\right\},\qquad e^{-\pi s}=e^{-as}\text{ and }\frac{1}{s^2+1}=F(s)\text{ in the formula,}$$

$$\Rightarrow\qquad a=\pi,\qquad\text{and}\quad f(t)=\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t,$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}=\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{s^2+1}\right\}=f(t-\pi)\mathcal{U}(t-\pi)=\sin(t-\pi)\mathcal{U}(t-\pi).$$

- $\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+4}\right\}$.

$$\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+4}\right\}=\mathcal{L}^{-1}\left\{e^{-2s}\frac{s}{s^2+4}\right\},\quad e^{-2s}=e^{-as}\text{ and }\frac{s}{s^2+4}=F(s)\text{ in the formula,}$$

$$\Rightarrow\qquad a=2,\qquad\text{and}\quad f(t)=\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}=\cos(2t),$$

$$\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+4}\right\}=\mathcal{L}^{-1}\left\{e^{-2s}\frac{s}{s^2+4}\right\}=f(t-2)\mathcal{U}(t-2)=\cos(2(t-2))\mathcal{U}(t-2)=\cos(2t-4)\mathcal{U}(t-2).$$

3. Use the operational property

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$$

to find the following Laplace transform or inverse Laplace transform.

(a) $\mathcal{L}\left\{e^{t}\sin(3t)\right\}$,

(b) $\mathcal{L}\left\{e^{2t}(t-1)^2\right\}$,

(c) $\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^4}\right\}$,

(d) $\mathcal{L}^{-1}\left\{\frac{5s}{(s-2)^2}\right\}$,

SOLUTION:

- $\mathcal{L}\left\{e^{t}\sin(3t)\right\}$. To use the formula,

$$e^{t} = e^{at} \quad \Rightarrow \quad a = 1,$$

and

$$f(t) = \sin(3t) \quad \Rightarrow \quad F(s) = \mathcal{L}\left\{f(t)\right\} = \mathcal{L}\left\{\sin(3t)\right\} = \frac{3}{s^2+9},$$

so the Laplace transform is

$$\mathcal{L}\left\{e^{t}\sin(3t)\right\} = F(s-1) = \frac{3}{(s-1)^2+9}.$$

- $\mathcal{L}\left\{e^{2t}(t-1)^2\right\}$.

$$e^{2t} = e^{at} \quad \Rightarrow \quad a = 2,$$

and

$$f(t) = (t-1)^2 = t^2 - 2t + 1 \quad \Rightarrow \quad F(s) = \mathcal{L}\left\{t^2\right\} - 2\mathcal{L}\left\{t\right\} + \mathcal{L}\left\{1\right\} = \frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s},$$

so the solution is

$$\mathcal{L}\left\{e^{2t}(t-1)^2\right\} = F(s-2) = \frac{2}{(s-2)^3} - \frac{2}{(s-2)^2} + \frac{1}{s-2}.$$

- $\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^4}\right\}$. We want to use the inverse of the property

$$\mathcal{L}^{-1}\left\{F(s-a)\right\} = e^{at}f(t),$$

so we should have

$$F(s-a) = \frac{1}{(s-4)^4}, \quad \Rightarrow \quad a = 4 \quad \text{and} \quad F(s) = \frac{1}{s^4},$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{t^3}{3!} = \frac{1}{6}t^3 \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{(s-4)^4}\right\} = \frac{1}{6}t^3e^{4t}.$$

- $\mathcal{L}^{-1}\left\{\frac{5s}{(s-2)^2}\right\}$. We want to use the inverse of the property

$$\mathcal{L}^{-1}\left\{F(s-a)\right\} = e^{at}f(t),$$

so

$$F(s-a) = \frac{5s}{(s-2)^2} = \frac{5(s-2)+10}{(s-2)^2} = \frac{5(s-2)}{(s-2)^2} + \frac{10}{(s-2)^2} = \frac{5}{s-2} + \frac{10}{(s-2)^2},$$

therefore we have

$$a = 2 \quad \text{and} \quad F(s) = \frac{5}{s} + \frac{10}{s^2} \quad \Rightarrow \quad f(t) = \mathcal{L}^{-1}\left\{\frac{5}{s} + \frac{10}{s^2}\right\} = 5 + 10\cdot\frac{t^1}{1!} = 5 + 10t,$$

the solution is

$$\mathcal{L}^{-1}\left\{\frac{5s}{(s-2)^2}\right\} = e^{2t}(5+10t).$$

4. Find the Laplace transform

$$\mathcal{L}\left\{t^3e^{10t}\right\}$$

by using the operational property

(a) $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.

(b) $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$.

SOLUTION:

- If we want to use property $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ for $\mathcal{L}\left\{t^3e^{10t}\right\}$, then

$$e^{10t} = e^{at} \quad \Rightarrow \quad a = 10,$$

and

$$f(t) = t^3 \quad \Rightarrow \quad F(s) = \mathcal{L}\left\{t^3\right\} = \frac{3!}{s^4} = \frac{6}{s^4},$$

so the Laplace transform is

$$\mathcal{L}\left\{t^3e^{10t}\right\} = \frac{6}{(s-10)^4}.$$

- If we want to use property $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$ for $\mathcal{L}\left\{t^3e^{10t}\right\}$, then

$$t^3 = t^n \quad \Rightarrow \quad n = 3,$$

and

$$f(t) = e^{10t} \quad \Rightarrow \quad F(s) = \mathcal{L}\left\{e^{10t}\right\} = \frac{1}{s-10},$$

so the Laplace transform is

$$\begin{aligned}
\mathcal{L}\left\{t^3e^{10t}\right\} = (-1)^3\frac{d^3}{ds^3}\left(\frac{1}{s-10}\right) &= (-1)^3[(s-10)^{-1}]''' \\
&= (-1)[(-1)(s-10)^{-2}]'' \\
&= (-1)[(-1)(-2)(s-10)^{-3}]' \\
&= (-1)(-1)(-2)(-3)(s-10)^{-4} \\
&= 6\cdot\frac{1}{(s-10)^4}.
\end{aligned}$$

5. Use Laplace Transform to solve the following I.V.P.

(a) $y' - y = te^t, \quad y(0) = 0;$

(b) $y' + y = f(t), \quad y(0) = 0$, where

$$f(t) = \begin{cases} 0, & 0 \le t < 1, \\ 5, & t \ge 1. \end{cases}$$

SOLUTION:

- $y' - y = te^t, \quad y(0) = 0.$

$$\begin{aligned}
sY(s) - y(0) - Y(s) &= \mathcal{L}\left\{te^t\right\}, \qquad a = 1, f(t) = t, F(s) = \frac{1}{s^2} \text{ in property } \mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a), \\
(s-1)Y(s) &= \frac{1}{(s-1)^2}, \\
Y(s) &= \frac{1}{(s-1)^3}, \\
y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\} = e^t\cdot\frac{t^2}{2!} = \frac{1}{2}t^2e^t.
\end{aligned}$$

DO NOT SKIP STEPS IN QUIZ.

the reasons that I skipped steps are:

– **there are enough examples showing how to apply the formula (similar to Q2(c))**;
– 2) I don't know if you know how to apply the formula without showing me the steps, so please show the steps!

(a) $y' + y = f(t), \quad y(0) = 0$, where

$$f(t) = \begin{cases} 0, & 0 \le t < 1, \\ 5, & t \ge 1. \end{cases}$$

$$sY(s) - y(0) + Y(s) = \mathcal{L}\{f(t)\}, \qquad f(t) = 5\mathcal{U}(t-1),$$
$$(s+1)Y(s) = 5\mathcal{L}\{\mathcal{U}(t-1)\} = 5 \cdot e^{-s} \cdot \frac{1}{s},$$
$$Y(s) = 5e^{-s} \cdot \frac{1}{s(s+1)},$$
$$Y(s) = 5e^{-s} \cdot \frac{1}{s} - 5e^{-s} \cdot \frac{1}{s+1}, \qquad \text{partial fraction decomposition},$$
$$y(t) = 5\mathcal{L}^{-1}\left\{e^{-s} \cdot \frac{1}{s}\right\} - 5\mathcal{L}^{-1}\left\{e^{-s} \cdot \frac{1}{s+1}\right\}, \qquad \text{use the inverse of } \mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s),$$
$$y(t) = 5\mathcal{U}(t-1) - 5e^{-(t-1)}\mathcal{U}(t-1).$$

PLEASE DO NOT SKIP STEPS IN QUIZ, SAME REASONS AS ABOVE.

6. Use Laplace Transform to solve the following the I.V.P.:

$$y'' + 9y = \cos(3t), \quad y(0) = 2, \quad y'(0) = 5.$$

Hint: use one of the operational property to get $\mathcal{L}\{t\sin(kt)\}$.

SOLUTION: Use the Laplace transform of 2nd order derivative,

$$\begin{aligned}
\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} &= \mathcal{L}\{\cos(3t)\} \\
(s^2Y(s) - sy(0) - y'(0)) + 9Y(s) &= \frac{s}{s^2+3^2} \\
s^2Y(s) - s \cdot 2 - 5 + 9Y(s) &= \frac{s}{s^2+9} \\
(s^2+9)Y(s) &= \frac{s}{s^2+9} + 2s + 5 \\
Y(s) &= \frac{s}{(s^2+9)^2} + \frac{2s+5}{s^2+9} \\
Y(s) &= \frac{s}{(s^2+9)^2} + 2 \cdot \frac{s}{s^2+9} + 5 \cdot \frac{1}{s^2+9} \\
y(t) &= \mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} + 2\cos(3t) + \frac{5}{3}\sin(3t)
\end{aligned}$$

Let us check out the hint: use the property $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$ to get $\mathcal{L}\{t\sin(kt)\}$, here $n = 1$ and $f(t) = \sin(kt)$, so $F(s) = \frac{k}{s^2+k^2}$,

$$\begin{aligned}
\mathcal{L}\{t\sin(kt)\} &= (-1)^1 \frac{d}{ds}F(s), \\
&= (-1) \cdot \left(\frac{k}{s^2+k^2}\right)' \\
&= (-1) \cdot \frac{0 \cdot (s^2+k^2) - k \cdot 2s}{(s^2+k^2)^2} \\
\mathcal{L}\{t\sin(kt)\} &= \frac{2ks}{(s^2+k^2)^2}
\end{aligned}$$

so the inverse Laplace transform $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\}$ in the solution is

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} = \frac{1}{2 \cdot 3}\mathcal{L}^{-1}\left\{\frac{2 \cdot 3 \cdot s}{(s^2+3^2)^2}\right\} = \frac{1}{6}t\sin(3t),$$

the solution of the initial value problem is

$$y(t) = \frac{1}{6} t \sin(3t) + 2\cos(3t) + \frac{5}{3}\sin(3t)$$