Week 8 Practice Solution

October 19, 2019

1. A large tank is filled with 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of 5 gal/min. The well-mixed solution is pumped out at the same rate.

   (a) Find the number $A(t)$ of pounds of salt in the tank at time t;

   (b) What is the concentration $c(t)$ of the salt in the tank at time t?

   SOLUTION: Actually we should answer part (b) first. To find the concentration $c(t)$ of salt in the tank at time t, it should be

$$\frac{\text{the amount of salt in tank at time t}}{\text{the amount of liquid in tank at time t}} = \frac{A(t) \text{ lb}}{(500+5t-5t) \text{ gallon}} = \frac{A(t)}{500} \text{ lb/gal},$$

   where the denominator $500 + 5t - 5t (= 500\text{gallon} + 5\text{gal/min} \times t\text{min} - 5\text{gal/min} \times t\text{min})$ is the amount of liquid in tank at time t calculated by: the initial amount of liquid + the amount of liquid being pumped in over t minutes - the amount of liquid being pumped out over t minutes. Now, let us solve part (a). To solve part (a), we need to set up a differential equation about $A(t)$,

$$\begin{aligned}\frac{dA(t)}{dt} &= \text{the rate of change of salt in tank at time t},\\ &= \text{the amount of salt changed in tank per minute},\\ &= \text{the amount of salt pumped in per minute - the amount of salt pumped out per minute}\\ &= 2 \text{ lb/gal } \times 5 \text{ gal/min} - \frac{A(t)}{500} \text{ lb/gal} \times 5 \text{ gal/min}\end{aligned}$$

   so the differential equation is $\frac{dA(t)}{dt} = 10 - \frac{A(t)}{100}$. On the other hand, the tank is filled with 500 gallon of pure water, so there is not salt in tank. We have the following initial value problem

$$\frac{dA(t)}{dt} = 10 - \frac{A(t)}{100}, \qquad A(0) = 0.$$

   To solve this initial value problem, let us use the formula for 1st order D.E.,

$$A'(t) + \frac{1}{100}A(t) = 10, \quad P(t) = \frac{1}{100}, \quad f(t) = 10, \quad \mu(t) = e^{\int P(t)dt} = e^{\int \frac{1}{100}dt} = e^{\frac{t}{100}},$$

$$\begin{aligned}A(t) &= \frac{\int 10e^{\frac{t}{100}}dt + C}{e^{\frac{1}{100}t}} = \frac{1000e^{\frac{t}{100}} + C}{e^{\frac{1}{100}t}} = 1000 + Ce^{-\frac{t}{100}},\\ A(0) = 0 \quad &\Rightarrow \quad 0 = 1000 + C \quad \Rightarrow \quad C = -1000,\end{aligned}$$

   so the amount of salt at time t is

$$A(t) = 1000 - 1000e^{\frac{t}{100}}.$$

2. A large tank contains 200 gallon of fuild in which 30 grams of salt is dissolved. Brine containing 1 gram of salt per gallon is then pumped into the tank at a rate of 4 gallon per minute, the well-mixed solution is pumped out at a rate of 6 gallon per minute. Find the number $A(t)$ of grams of salt in the tank at time t.

SOLUTION: Same method with the previous question. Let use find the concentration $c(t)$ of salt in tank at time t.

$$c(t) = \frac{A(t)}{200(\text{initial amount}) + 4t(\text{amount pumped in over t minutes}) - 6t(\text{amount pumped out over t minutes})}$$
$$= \frac{A(t)}{200-2t} \text{ gram/gal},$$

then the differential equation can be given by

$$\frac{dA(t)}{dt} = 1\text{gram/gal} \times 4\text{gal/min} - \frac{A(t)}{200-2t} \text{ gram/gal} \times 6\text{gal/min} = 4 - \frac{6}{200-2t}A(t),$$

on the other hand, there were 30 grams of salt is dissolved, it means $A(0) = 30$. So the entire question can be explained in terms of the i.v.p.

$$A'(t) = 4 - \frac{3}{100-t}A(t), \quad A(0) = 30.$$

Now let us solve it

$$A' + \frac{3}{100-t}A = 4, \quad P(t) = \frac{3}{100-t}, \quad f(t) = 4, \quad \mu(t) = e^{\int \frac{3}{100-t}dt} = e^{-3\ln|100-t|} = \left(e^{\ln|100-t|}\right)^{-3} = \frac{1}{(100-t)^3},$$

$$A(t) = \frac{\int \frac{4}{(100-t)^3}dt + C}{\frac{1}{(100-t)^3}} = (100-t)^3[4 \cdot \left(\frac{1}{-2}\right)(100-t)^{-2} + C] = -2(100-t) + C(100-t)^3,$$

use the initial condition $A(0) = 30$,

$$30 = -2 \cdot 100 + C \cdot 100^3 \quad \Rightarrow \quad C = 2.3 \times 10^{-3},$$

so the solution of this question is

$$A(t) = -2(100-t) + 2.3 \times 10^{-3}(100-t)^3.$$

3. A small metal bar, whose initial temperature was 20^oC, is dropped into a large container of boiling water (100^oC). How long will it take the bar to reach 90^oC if it is known that its temperature increases 2^oC in 1 second? (The rate of temperatire change is proportional to the difference between its temperature and the surrounding temperature).

   SOLUTION: Let us use the function $T(t)$ to denote the desired temperature - the temperature of the metral bar at time t. The fact that the metal bar has initial temperature 20 degree means $T(0) = 20$ (initial condition), and the fact that its temperature increases 2 degree in 1 second means $T(1) = 20 + 2 = 22$ (what we normally call boundary condition). Now let us set up the differential equation due the 'proportional' property (Newton's Law of cooling/warming),

$$\frac{dT(t)}{dt} = k(T(t) - 100), \quad \text{where 100 is the temperature of water (surrounding environment)},$$

   so the question can be interpreted as the following initial value problem (with boundary condition), assume the proportional constant $k > 0$,

$$\begin{cases} \frac{dT}{dt} &= -k(T-100), \\ T(0) &= 20, \\ T(1) &= 22, \end{cases}$$

   once we solve $T(t)$, we are able to find how will it take to warm up to 90 degree ($t =?$ such that $T(t) = 90$). The differential equation is separable

$$\frac{dT}{(T-100)} = -kdt, \quad \Rightarrow \quad \ln|T-100| = -kt + C,$$

the condition $T(0) = 20$ gives

$$\ln|20 - 100| = -k \cdot 0 + C, \quad \Rightarrow \quad C = \ln 80,$$

and the condition $T(1) = 22$ gives

$$\ln|22 - 100| = -k \cdot 1 + \ln 80, \quad \Rightarrow \quad k = \ln 80 - \ln 78 (= \ln \frac{80}{78} \approx 0.0253),$$

so the solution for $T(t)$ is

$$\ln|T(t) - 100| = -\ln \frac{80}{78} \cdot t + \ln 80 \quad \text{or, equivalently,}$$

$$|T(t) - 100| = e^{-\ln \frac{80}{78} \cdot t + \ln 80}, \quad -(T(t) - 100) = e^{\ln 80} \cdot \left(e^{\ln \frac{80}{78}}\right)^{-t}, \quad T(t) = 100 - 80\left(\frac{80}{78}\right)^{-t}.$$

Therefore, when $T(t) = 90$, we have

$$\ln|90 - 100| = -\ln \frac{80}{78} \cdot t + \ln 80, \quad \Rightarrow \quad t = \frac{\ln 80 - \ln 10}{\ln(\frac{80}{78})} = \ln 8 / \ln(\frac{80}{78}) \approx 82.13 \,\text{sec}.$$

It will take 82.13 seconds to warm up the metral bar to 90 degree. (If the question asks for the temperature of the metal bar after a certain amount of time, it will be easier to use the solution in the form of $T(t) = 100 - 80\left(\frac{80}{78}\right)^{-t}$).

4. Suppose a student carring a flu virus returns to an isolated college campus of 2000 students. If it is assumed that the rate at which the virus spread is proportional not only to the number $P(t)$ of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 3 days there are 20 students infected.

SOLUTION: Let us set up a differential equation regards the number $P(t)$ of infected students. The relation between the rate at which the virus spread and the population of infected students is "the rate at which the virus spread is proportional not only to the number $P(t)$ of infected students but also to the number of students not infected". The rate of spread is $\frac{dP(t)}{dt}$, the number of students infected is $P(t)$, and the number of students not infected $2000 - P(t)$ (since there are 2000 students in total). So the differential equation is

$$\frac{dP}{dt} = kP(2000 - P).$$

On the other side, there are 20 students infected after 3 days means that $P(3) = 20$, and when the student who carrys the virus stepped in the campus, this student is the only one of the 2000 students who is infected, so $P(0) = 1$. Now the situation can be interpreted as

$$\begin{cases} \frac{dP}{dt} & = kP(2000 - P), \\ P(0) & = 1, \\ P(3) & = 20, \end{cases}$$

and we want to know $P(6) = ?$. To solve it, the differential equation is separable,

$$\frac{dP}{P(2000 - P)} = kdt, \quad \frac{1}{2000}(\frac{1}{P - 2000} - \frac{1}{P})dP = kdt, \quad \frac{1}{2000}(\ln|P - 2000| - \ln|P|) = k_0 t + C_0,$$

$$\ln\left|\frac{P - 2000}{P}\right| = 2000k_0 t + 2000C_0, \quad \left|\frac{P - 2000}{P}\right| = e^{2000k_0 \cdot t} \cdot e^{2000C_0} = Ce^{kt},$$

since $P < 2000$, the fraction $\frac{P-2000}{P} < 0$, so we have

$$-\frac{P - 2000}{P} = Ce^{kt}, \quad \Rightarrow \quad P(t) = \frac{2000}{1 + Ce^{kt}}.$$

The initial condition $P(0) = 1$ implies

$$1 = \frac{2000}{1 + C \cdot 1}, \quad \Rightarrow \quad C = 1999,$$

the boundary condition $P(3) = 20$ implies

$$20 = \frac{2000}{1 + 1999e^{k \cdot 3}} \quad \Rightarrow \quad e^k = \left(\frac{99}{1999}\right)^{\frac{1}{3}},$$

so the solution is

$$P(t) = \frac{2000}{1 + 1999\left(\frac{99}{1999}\right)^{\frac{t}{3}}}, \quad P(6) = \frac{2000}{1 + 1999\left(\frac{99}{1999}\right)^{\frac{6}{3}}} \approx 339.$$

There will be 339 students infected.

5. Two chemicals A and B are combined to form a chemical C. The rate, or velocity, of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially, there are 40 grams of A and 50 grams of B, and for each gram of B, 2 grams of A is used. Let $X(t)$ be thhe amount of C is formed in $t$minues. Set up a D.E. of $X(t)$ and then solve it.

SOLUTION: 'for each gram of B, 2 grams of A is used' means

$$1\text{g B} + 2\text{g A} = (1+2)\text{g C} = 3\text{g C},$$

so for each gram of C being formed, the amount of B and A being used is

$$\frac{1}{3}\text{g B} + \frac{2}{3}\text{g}A = 1\text{g C},$$

so when $X(t)$ grams of C is formed, the amount of B and A will be used is

$$\frac{1}{3}X(t)\ \text{g B} + \frac{2}{3}X(t)\ \text{g A} = X(t)\ \text{g C}.$$

There are 50 grams of B and 40 grams of A initially, so the amount of B and A not converted is $(50 - \frac{1}{3}X)$ and $(40 - \frac{2}{3}X)$, respectively. The D.E. is

$$\frac{dX}{dt} = k(50 - \frac{1}{3}X)(40 - \frac{2}{3}X), \quad \text{with initial condition} \quad X(0) = 0.$$

Let us solve the differential equation, it is separable,

$$\frac{dX}{(50 - \frac{1}{3}X)(40 - \frac{2}{3}X)} = kdt,$$

$$-\frac{1}{60}(\frac{1}{50 - \frac{1}{3}X} - \frac{2}{40 - \frac{2}{3}X})dX = kdt, \quad \text{partial fraction decomposition,}$$

$$-\frac{1}{60}\left[\frac{1}{-\frac{1}{3}}\ln|50 - \frac{1}{3}X| - \frac{2}{-\frac{2}{3}}\ln|40 - \frac{2}{3}X|\right] = kt + C,$$

$$\frac{1}{20}\ln\frac{150 - X}{120 - 2X} = kt + C, \quad \text{simplified, personal choice,}$$

use the initial condition

$$\frac{1}{20}\ln\frac{150 - 0}{120 - 2 \cdot 0} = k \cdot 0 + C, \quad \Rightarrow \quad C = \frac{\ln\frac{5}{4}}{20},$$

so the solution is

$$\ln\frac{150 - X}{120 - 2X} = kt + \ln\frac{5}{4}.$$

Note: we need one more boundary condition to solve for the proportional constant, it is not provided in this questions, we can just stop here.

6. Find the charge $q(t)$ on the capacitor in an LRC-series circuit when $L = 0.25$ h, $R = 10\,\Omega$, $C = 0.001$ f, $E(t) = 0$, $q(0) = q_0$ C, and $i(0) = 0$.

SOLUTION: Kirchhoff's second law

$$Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = E(t),$$

and $q'(t) = i(t)$. So the question can be interpreted as

$$\begin{cases} 0.25q'' + 10q' + 1000q & = 0, \\ q(0) & = q_0 = 10, \quad \text{let us assume } q_0 = 10 \\ q'(0) = i(0) & = 0. \end{cases}$$

It is a second order D.E., we can only use Laplace transform to solve currently,

$$\begin{aligned} 0.25\mathcal{L}\{q''\} + 10\mathcal{L}\{q'\} + 1000\mathcal{L}\{q\} &= 0, \\ 0.25(s^2Q(s) - s \cdot 10 - 0) + 10(sQ(s) - 10) + 1000Q(s) &= 0, \\ (s^2 + 40s + 4000)Q(s) &= 10s + 400, \\ Q(s) &= \frac{10s + 400}{(s+20)^2 + 60^2}, \\ Q(s) &= \frac{10(s+20) + 200}{(s+20)^2 + 60^2}, \\ Q(s) &= 10\frac{(s+20)}{(s+20)^2 + 60^2} + \frac{200}{(s+20)^2 + 60^2}, \\ q(t) &= 10e^{-20t}\cos(60t) + \frac{200}{60}e^{-20t}\sin(60t), \end{aligned}$$

where the last step uses $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$, some steps are simple algbra simplification.