Week 9 Practice

October 27, 2019

1. Determine whether the given set of functions is linearly independent or not:

   (a) $f_1(x) = x, \quad f_2(x) = x^2, \quad f_3(x) = 4x - 3x^2$;

   (b) $f_1(x) = e^x, \quad f_2(x) = e^{-x}$;

   (c) $f_1(x) = 1, \quad f_2(x) = \sin x, \quad f_3(x) = \cos x$;

   (d) $f_1(x) = x, \quad f_2(x) = e^x, \quad f_3(x) = xe^x$;

   (e) $f_1(x) = \cos^2 x - \sin^2 x,\ f_2(x) = \sin^2 x,\ f_3(x) = 1$.

   SOLUTION:

   (a) Linearly dependent. It is not hard to see that $f_3 = 4f_1 - 3f_2$, so it is linearly dependent.

   (b) Linearly independent. To show it is linearly independent, we need to check its determinant of the Wronskian matrix is not 0.

   $$W(f_1, f_2) = \det \begin{pmatrix} e^x & e^{-x} \\ e^x & (e^{-x})' \end{pmatrix} = \det \begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix} = e^x \cdot (-e^{-x}) - e^x \cdot e^{-x} = -e^0 - e^0 = -2 \neq 0.$$

   (c) Linearly independent. To show it is linearly independent, we need to check its determinant of the Wronskian matrix is not 0.

   $$\begin{aligned} W(f_1, f_2, f_3) &= \det \begin{pmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{pmatrix} \\ &= 1 \cdot \begin{vmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{vmatrix} - \sin x \cdot \begin{vmatrix} 0 & -\sin x \\ 0 & -\cos x \end{vmatrix} + \cos x \cdot \begin{vmatrix} 0 & \cos x \\ 0 & -\sin x \end{vmatrix} \\ &= 1 \cdot [-\cos^2 x - \sin^2 x] = -1 \neq 0. \end{aligned}$$

   (d) Linearly independent. To show it is linearly independent, we need to check its determinant of the Wronskian matrix is not 0.

   $$\begin{aligned} W(f_1, f_2, f_3) &= \det \begin{pmatrix} x & e^x & xe^x \\ 1 & e^x & e^x + xe^x \\ 0 & e^x & 2e^x + xe^x \end{pmatrix} \\ &= x \cdot \begin{vmatrix} e^x & e^x + xe^x \\ e^x & 2e^x + xe^x \end{vmatrix} - e^x \cdot \begin{vmatrix} 1 & e^x + xe^x \\ 0 & 2e^x + xe^x \end{vmatrix} + xe^x \begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix} \\ &= x \cdot (2e^{2x} + xe^{2x} - e^{2x} - xe^{2x}) - e^x \cdot (2e^x + xe^x - 0) + xe^x \cdot (e^x - 0) \\ &= xe^{2x} - 2e^{2x} - xe^{2x} + xe^{2x} = xe^{2x} - 2e^{2x}. \end{aligned}$$

   (e) Linearly dependent. It is not hard to see that $f_1 + 2f_2 = f_3$. ($\cos^2 x = f_1 + f_2$, since $\cos^2 x + \sin^2 x = 1$, so $f_1 + f_2 + f_2 = f_3$.)

2. Find a particular solution of $y'' + 2y = 8x + 5$ by answering the following questions.

   (a) Find a particular solution of $y'' + 2y = 1$ by inspection;

   (b) Find a particular solution of $y'' + 2y = 4x$ by inspection;

(c) Use superposition principle to find a particular solution of $y'' + 2y = 8x + 5$.

SOLUTION:

(a) Ignore the higher order derivative, let $2y = 1$, we have $y_{p_1} = \frac{1}{2}$ will be a particular solution of $y''+2y = 1$. (Becuase $y''_{p_1} = (1/2)'' = 0$).

(b) Ignore the higher order derivative, let $2y = 4x$, we have $y_{p_2} = 2x$ is a particular solution of $y''+2y = 4x$. (Because $y''_{p_2} = (2x)'' = (2)' = 0$).

(c) Since $y'' + 2y = 2 \cdot 4x + 5 \cdot 1$, so $y_p = 2 \cdot y_{p_2} + 5 \cdot y_{p_1} = 2 \cdot 2x + 5 \cdot \frac{1}{2} = 4x + \frac{5}{2}$ is a particular solution of $y'' + 2y = 8x + 5$.

3. Find a particular solution of $y'' - 3y' + 4y = 24x - 8$. (Hint: similar to question 2.)

SOLUTION:

On the right hand side, the highest power term of x is $24x$, since the second order derivative of $24x$ is zero ($(24x)'' = (24)' = 0$), so we can ignore the second order derivative on the left y''. To find a particular solution, let us assume it has the form $y_p = Ax + B$. To determine the values for A and B, we have

$$y''_p - 3y'_p + 4y_p = 24x - 8,$$
$$0 - 3 \cdot A + 4Ax + 4B = 24x - 8,$$

so $4A = 24$, i.e., $A = 6$, and $4B - 3A = -8 \Rightarrow B = \frac{1}{4}(-8 + 3 \cdot 6) = \frac{5}{2}$. A particular solution is $y_p = 6x + \frac{5}{2}$.

4. Find the general solution of the given higher-order differential equation or solve the following IVP:

(a) $y''' - 4y'' - 5y' = 0$;

(b) $\frac{d^2y}{dx^2} - 36y = 0$;

(c) $4y'' - 4y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 5$;

(d) $y''' + 12y'' + 36y' = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 7$.

SOLUTION: All the differential equations are homogeneous linear differential equations with constant coefficients.

(a) $m^3 - 4m^2 - 5m = 0$, so we have $m(m - 5)(m + 1) = 0$, we have three distinct solutions $m_1 = 0, m_2 = 5, m_3 = -1$, so the general solution is $y = C_1e^{0x} + C_2e^{5x} + C_3e^{-x} = C_1 + C_2e^{5x} + C_3e^{-x}$.

(b) $m^2 - 36 \cdot m^0 = m^2 - 36 = 0$, we have $(m+6)(m-6) = 0$, we have two distinct solutions $m_1 = -6, m_2 = 6$, so the general solution is $y = C_1e^{-6x} + C_2e^{6x}$.

(c) $4m^2 - 4m - 3 \cdot m^0 = 4m^2 - 4m - 3 = 0$, we have $(2m + 1)(2m - 3) = 0$, two distinct solutions $m_1 = -\frac{1}{2}, m_2 = \frac{3}{2}$. the general solution is $y = C_1e^{-\frac{1}{2}x} + C_2e^{\frac{3}{2}x}$. Let us use the initial conditions to find values for C_1 and C_2.

$$\begin{array}{llll} y(0) = 1 & \Rightarrow & C_1 \cdot e^0 + C_2 \cdot e^0 = 1 & \Rightarrow \quad C_1 + C_2 = 1, \\ y'(0) = 5 & \text{calculate } y' \text{ first,} & y'(x) = -\frac{1}{2}C_1e^{-\frac{1}{2}x} + \frac{3}{2}C_2e^{\frac{3}{2}x} & \\ & \Rightarrow & 5 = -\frac{1}{2}C_1 + \frac{3}{2}C_2 & \Rightarrow \quad -\frac{1}{2}C_1 + \frac{3}{2}C_2 = 5, \end{array}$$

so the second equation gives $-C_1+3C_2 = 10 \Rightarrow -C_1+3C_2+C_1+C_2 = 10+1 \Rightarrow 4C_2 = 11$, so $C_2 = 11/4$, $C_1 = 1 - C_2 = 1 - 11/4 = -7/4$. The solution of this IVP is

$$y = -\frac{7}{4}e^{-\frac{1}{2}x} + \frac{11}{4}e^{\frac{3}{2}x}.$$

(d) $m^3 + 12m^2 + 36m = 0$, we have $m(m + 6)(m + 6) = 0$, $m_1 = 0$, and a repeated root $y_2 = y_3 = -6$, the fundamental set of solution is $\{e^{0x}, e^{-6x}, xe^{-6x}\} = \{1, e^{-6x}, xe^{-6x}\}$. The general solution is

$$y = C_1 + C_2e^{-6x} + C_3xe^{-6x},$$

$$y' = -6C_2e^{-6x} + C_3(e^{-6x} - 6xe^{-6x}) = (C_3 - 6C_2)e^{-6x} - 6C_3xe^{-6x},$$
$$y'' = -6(C_3 - 6C_2)e^{-6x} - 6C_3(e^{-6x} - 6xe^{-6x}) = -6(2C_3 - 6C_2)e^{-6x} + 36C_3xe^{-6x},$$

the initial conditions lead to

$$\begin{array}{rcccc} y(0) = 0 & \Rightarrow & 0 = C_1 + C_2 + C_3 \cdot 0 & \Rightarrow & C_1 + C_2 = 0, \\ y'(0) = 1 & \Rightarrow & 1 = (C_3 - 6C_2) \cdot 1 - 6C_3 \cdot 0 & \Rightarrow & -6C_2 + C_3 = 1, \\ y''(0) = 7 & \Rightarrow & 7 = -6(2C_3 - 6C_2) \cdot 1 + 36C_3 \cdot 0 & \Rightarrow & 36C_2 - 12C_3 = 7, \end{array}$$

so we have,

$$6(-6C_2 + C_3) + 36C_2 - 12C_3 = 6 \cdot 1 + 7 \quad \Rightarrow \quad -6C_3 = 13 \quad \Rightarrow \quad C_3 = -13/6,$$

$$C_2 = (C_3 - 1)/6 = (-13/6 - 1)/6 = -19/36 \quad \Rightarrow \quad C_1 = -C_2 = 19/36$$

the solution is

$$y = 19/36 - 19/36e^{-6x} - 13/6xe^{-6x}.$$